The non-isomorphism of certain continuous rings.

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This manuscript, one of several written by von Neumann in 1935–37, is published now, after his death. The paper begins with the following introductory remarks by Irving Kaplansky.

"By a continuous geometry \( L \) we mean here an irreducible one, in the infinite case. There exists a unique regular ring \( R \) coordinatizing \( L \), and \( R \) is called a continuous ring. It is a simple ring with unit, and its center is a field \( Z \). Starting with an arbitrary field \( Z \), von Neumann had given a purely algebraic construction of a continuous geometry \( L_\infty(\mathbb{Z}) \) whose coordinatizing continuous ring \( Z_\infty \) has center \( Z \). Now in case \( Z \) is the field of complex numbers, there is another way to get a continuous ring with center \( Z \), namely from a factor \( M \) of type II\(_1\). The corresponding continuous ring \( U \) is obtained by adjoining to \( M \) suitable unbounded operators. In this paper von Neumann shows that the algebraic \( Z_\infty \) is never isomorphic to a ring \( U \) derived from a factor of type II\(_1\)."

Let \( R \) be any continuous ring with centre \( Z \) consisting of all complex numbers, e.g., \( Z_\infty \) or \( U \). A family of ring elements \( s_{ijl} \), \( l = 0, 1, 2, \ldots, i, j = 1, \ldots, n_l \) (with \( n_0 = 1 \), \( n_l = n_{l-1}q_l \) for \( l > 0 \) and each \( q_l \) an integer more than 1), is called a continuous set of matrix units if (i) for each fixed \( l \) the \( s_{ijl} \) are a set of matrix units, in particular \( s_{11l} = 1 \), and (ii) \( s_{ijl}^{-1} = \sum t s_{(i-1)q_l+t,(j-1)q_l+t} \). Let \( \gamma \) denote the set of all finite linear combinations, with coefficients in \( Z \), of the \( s_{ijl} \).

Von Neumann calls a ring element \( \pi \) continuous with respect to the \( s_{ijl} \) if for each \( l \) there are distinct rational numbers \( \rho_{il} \), \( i = 1, \ldots, n_l \), such that

\[
 s_{1il}(\pi + \rho_{il}s_{11l})s_{1il}^{-1} = s_{ii}l\pi = \pi s_{ii}l.
\]

He proves that such an \( \pi \) has rank metric distance from every element in \( \gamma \) equal to 1 so that such an \( \pi \) cannot be in the closed set (rank metric topology) determined by \( \gamma \).

Now the definition of \( Z_\infty \) shows that for a suitable continuous set of matrix units, \( Z_\infty \) coincides with the closed set determined by \( \gamma \). Thus there is no such \( \pi \) in \( Z_\infty \) with respect to this particular continuous set of matrix units.

On the other hand, for arbitrary continuous set of matrix units in a ring \( U \) derived from a factor of type II\(_1\), von Neumann constructs an \( \pi \) which is continuous with respect to these matrix units. It follows that \( Z_\infty \) is not ring isomorphic to any such \( U \).

Typographical errors are as follows. (1) Page 486, end of line 18: replace \((\pi)\) by \((\pi)_r\). (2) Page 486, line 3 from bottom should read: “We prove next \(((\pi_1 + \rho_{il}s_{ii}l)\pi)_r; i = 1, \ldots, n_l \) \perp, where \( \perp \) indicates that the ideal is taken in \( \mathfrak{R}(s_{11l}) \).” (3) Page 486, line 2 from bottom (twice) and page 487, lines 12 (twice), 13, 15 (twice): replace “\( \pi \)” by “\( \pi_1 \)”. (4) Page 487, line 16: replace “\( \pi \)” by “\( \pi_1 \)”. (5) Page 488, line and page 489 lines 4, 5: replace “\( \mathfrak{R}(s_{11l})^n \)” by “\( \mathfrak{R} \)”. 

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