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U represent the set of all \( G \) constructed using the elements of \( \text{app. finite} \). It is shown that any given but a sketch may suffice. If \( G \) interesting, however, is the construction of new examples. Complete details cannot be fundamental group of app. finite rings exhausts the positive numbers. Various examples \( 1 \) is also in case (II \( a \) ... 

\[ \alpha A \] is independent of the scaffold space \( H \). It is essential in this study to determine purely algebraic properties, properties independent of the notion of isomorphism. An algebraic ring isomorphism of \( M_1 \) and \( \mathbb{M} \) is a 1–1 correspondence leaving \( 1, \alpha A, A^*, A + B \) and \( AB \) invariant. A spatial isomorphism of \( M_1 \) and \( M_2 \) is an isometric mapping of \( \mathcal{H}_1 \) on \( \mathcal{H}_2 \) (the spaces over which \( M_1 \) and \( M_2 \) are defined) which transforms \( M_1 \) into \( M_2 \). The paper considers two questions. (1) When are \( M_1 \) and \( M_2 \) algebraically isomorphic? (2) When does algebraic imply spatial isomorphism?

It is essential in this study to determine purely algebraic properties, properties independent of the scaffold space \( \mathcal{H} \). A fundamental conclusion based on previous results of von Neumann [cf. the preceding review; MR0009095] is that eight types of closure (some algebraic, some spatial in character) are equivalent. This settles a problem of topology in a happy manner; the notion of subring is purely algebraic. If \( M \) is a ring, \( \mathbb{M} \) represents the set of rings algebraically isomorphic to \( M \). Fundamental to the paper is the concept \( \mathbb{M}^p \), where \( p \) assumes certain values depending on \( \mathbb{M} \), \( 0 < p \leq \infty \). The development begins with integral \( p \). Consider a matrix of operators \( \langle A_{ts} \rangle \), \( t, s = 1, 2, \ldots, p, A_{ts} \in \mathbb{M} \); the totality of these matrices forms \( \mathbb{M}^p \). Having considered integral \( p \), certain rational and ultimately (for the case (I)) real values are introduced; these behave like numerical exponents. The values \( \theta \) for which \( \mathbb{M}^\theta = \mathbb{M} \) form a multiplicative group \( \mathfrak{B}(\mathbb{M}) \), the fundamental group of \( \mathbb{M} \). Factors \( \mathbb{M} \) and \( \mathbb{N} \) are called commensurable if \( \mathbb{M} = \mathbb{N}^\theta \) or \( \mathbb{N} = \mathbb{M}^\theta \) with \( 0 < \theta \leq \infty \). As this is an equivalence, commensurable \( \mathbb{M} \) are thereby united into a genus \( \mathbb{M} \). All rings which are spatially isomorphic form a spatial type \( \mathbb{M} \). With these notions problem (2) is answered: for factors in the cases (I) or (II), \( \mathbb{M} \) is completely determined by \( \mathbb{M}, \mathbb{M}^\theta \) and a certain dimension ratio which cannot here be defined.

An important contribution in this paper is the concept of approximately (app.) finite ring. Four definitions are given whose equivalence is subsequently established. Probably the most suggestive is this. Let \( p_1, p_2, \ldots \to \infty \) be integers with \( p_n \) a divisor of \( p_{n+1} \). Let \( \mathbb{N}_n \) be rings in the case (I \( p_n \)), \( \mathbb{N}_1 \subset \mathbb{N}_2 \subset \cdots \). Then \( \mathbb{M} = \mathbb{R}(\mathbb{N}_n; n = 1, 2, \cdots) \) is app. finite. It is shown that any \( \mathbb{M} \) in case (II) contains an app. finite subring which is also in case (II). Also, all app. finite rings are algebraically isomorphic. Finally, the fundamental group of app. finite rings exhausts the positive numbers. Various examples previously introduced are discussed from the point of view of app. finiteness. Most interesting, however, is the construction of new examples. Complete details cannot be given but a sketch may suffice. If \( \mathfrak{B} \) is a countably infinite group, a Hilbert space \( \mathcal{H} \) is constructed using the elements of \( \mathfrak{B} \) as indices. The operations \( a \to a a_0, a \to a_0^{-1} a, a \to a^{-1} \) in \( \mathfrak{B} \) define in \( \mathcal{H} \) unitary operations \( U_{a_0}, V_{a_0}, W \) with \( V_{a_0} = W U_{a_0} W \). If \( I \) and \( J \) represent the set of all \( U_{a_0} \) and \( V_{a_0} \), respectively, \( \mathbb{R}(I) = J^\prime, \mathbb{R}(J) = I^\prime \). These are factors
if the class \(\{c^{-1}ac; c \in G\}\) of conjugates of \(a\) is infinite for every \(a \neq 1\). Finally they are approximately finite if \(\mathcal{G}\) is the set sum of finite groups \(\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \cdots\). This example shows how close these studies are to the theory of group representation.

A new invariant \(\Gamma\) of approximately finite rings is introduced. Using a group background, a factor \(M\) devoid of property \(\Gamma\) is constructed. Thus not all factors in case \((\Pi_1)\) are algebraically isomorphic. An example is given of two factors \((\Pi_1)\) which are not isomorphic and yet each is isomorphic to a subset of the other. This is the present state of knowledge on question (1).

The paper is very carefully cross-indexed; all necessary details are introduced. Thus the results of this series are readily accessible to a moderately initiated reader.

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