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$L^2$ harmonic forms and a conjecture of Dodziuk-Singer.

From the text: “Let $M^n$ be a complete simply connected Riemannian manifold of sectional curvature $K_M$ satisfying $-a^2 \leq K_M \leq -1$, $a \geq 1$. Let $\mathcal{H}^p_2(M^n)$ denote the space of $L^2$ harmonic $p$-forms on $M$, i.e., $p$-forms $\omega \in \Lambda^p(M^n)$ such that $\Delta \omega = 0$, $\int_{M^n} \omega \wedge \ast \omega = \int_{M^n} |\omega|^2 dV < \infty$. It is clear that $\mathcal{H}^p_2(M^n)$ is naturally isomorphic to $\mathcal{H}^{n-p}_2$ under the Hodge $\ast$ operator, and $\mathcal{H}^0_2(M^n) = 0$. Further, it is known that $\mathcal{H}^p_2(M^n)$ naturally injects into the $L^2$-cohomology of $M^n$. Dodziuk and Singer have conjectured that $\mathcal{H}^p_2(M^n) = 0$ if $p \neq n/2$ and $\dim \mathcal{H}^{n/2}_2 = \infty$ if $n$ is even. An affirmative solution of this conjecture implies, by means of the $L^2$ index theorem for regular covers of M. F. Atiyah [Colloquium on analysis and topology in honor of Henri Cartan (French) (Orsay, 1974), 43–72, Astérisque, 32–33, Soc. Math. France, Paris, 1976; MR0420729], a positive solution of the well-known Hopf conjecture: If $M^{2m}$ is a compact manifold of negative sectional curvature, then $(-1)^m \chi(M^n) > 0$. J. Dodziuk [Proc. Amer. Math. Soc. 77 (1979), no. 3, 395–400; MR0545603] has proved the $L^2$ form conjecture for rotationally symmetric metrics—in particular for the space forms $H^n(-a^2)$ of curvature $-a^2$. H. Donnelly and F. Xavier [Amer. J. Math. 106 (1984), no. 1, 169–185; MR0729759] have recently obtained results in case the curvature of $M^n$ is sufficiently pinched: They showed that $\mathcal{H}^p_2(M^n) = 0$ if $0 < p < (n-1)/2$ and $a < (n-1)/2p$.

“In this note, we outline the construction of counterexamples to the $L^2$ form conjecture, in every dimension and degree except the middle. Our main result is the following. Theorem: For any $n \geq 2$, $0 < p < n$ and $a > |n-2p|$, with $a \geq 1$, there exist complete simply connected Riemannian manifolds $M^n$ with $-a^2 \leq K_M \leq -1$ such that $\dim \mathcal{H}^p_2(M^n) = \infty$.”

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