The author considers a dynamical system defined on a 2-dimensional torus $T^2$ by the system of differential equations

$$\frac{dx}{dt} = A(x, y), \quad \frac{dy}{dt} = B(x, y),$$

and possessing an invariant integral $I(g) = \iint_{T^2} U(x, y) \, dx \, dy$, where $A$, $B$ and $U$ are univalued, analytic periodic functions of $x$ and $y$ with period $2\pi$. Here $x$ and $y$ are real coordinates mod $2\pi$, $A^2 + B^2 > 0$, $U > 0$ on the whole of $T^2$. It is then known [Nemyckii and Stepanov, Qualitative theory of differential equations, 2nd. ed., Gostehizdat, Moscow-Leningrad, 1949; for a review of the 1st ed. see MR0029483] that there exists an analytic transformation of coordinates which transforms the system (1) into the system

$$\frac{dx}{dt} = \frac{1}{F(x, y)}, \quad \frac{dy}{dt} = \frac{\gamma}{F(x, y)}$$

with an integral invariant $I(g) = \iint_{T^2} F(x, y) \, dx \, dy$ where $\gamma$ is a constant.

The following theorem is asserted. Theorem 1. If there exist constants $c > 0$ and $h > 0$ such that for all positive integers $m$ and $n$

$$|m - n\gamma| \geq ch^n,$$

then there exists an analytic transformation of coordinates which transforms the system (2) into the system

$$\frac{du}{dt} = \lambda_1, \quad \frac{dv}{dt} = \lambda_2,$$

where $\lambda_1, \lambda_2$ are constants and $\lambda_2 = \gamma \lambda_1$ and with the integral invariant $I(g) = K \iint_{T^2} du \, dv$. Condition (i) is fulfilled for every $\gamma$ except for a set of Lebesgue measure zero ($c$ and $h$ depend on $\gamma$). It follows that system (1) has a pure point spectrum with analytic proper functions.

For those irrational numbers which do satisfy (i) the author states: Theorem 2. Each of the following conditions is possible for a suitable choice of $\gamma$ and $F(x, y)$: The system (2) can be transformed into (3) by (I) an infinitely differentiable but not analytic transformation, (II) a $k$-differentiable but not $(k+1)$-differentiable transformation, (III) an everywhere-discontinuous transformation; and (IV) the system (2) cannot be transformed into (3) at all. In (I), (II) and (III) the original system (1) has a pure point spectrum but the proper functions are respectively not analytic, not $k+1$ differentiable and everywhere discontinuous. The conjecture is made that in (IV) the spectrum is necessarily continuous but only a considerably weaker result is proved. In all statements related to Theorem 2 the notions of analyticity, differentiability, etc. are interpreted modulo sets of Lebesgue measure zero. The method of obtaining the system (3) from (2) is obtained and discussed.

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