Let $\mathcal{S}$ be a separable and compact space with a Lebesgue measure possessing the usual properties, and $\mathfrak{C}$ a transitive group of motions $s: P \rightarrow sP$ in $\mathcal{S}$. A function $F(P, Q, \cdots)$ in $\mathfrak{C}$ is called group invariant if $F(sP, sQ, \cdots) \equiv F(P, Q, \cdots)$ for all $s \in \mathfrak{C}$. The main subject of this paper is the study of those group-invariant metrics $\rho(P, Q)$ in $\mathcal{S}$ (generating the given topology of $\mathcal{S}$, possibly with identifications) with which $\mathcal{S}$ can be imbedded isometrically into Hilbert space, that is, the Hilbert distances in $\mathcal{S}$. Following K. Menger [Math. Ann. 103, 466–501 (1930)] and I. J. Schoenberg [Ann. of Math. (2) 41, 715–726 (1940); cf. MR0002903], a connection is established between these distances and the positive functions of $\mathcal{S}$, that is, those continuous and group-invariant functions $f(P, Q)$ for which

$$\sum_{i,j=1}^{n} f(P_i, P_j) \rho_i \rho_j \geq 0$$

for all finite systems $P_i, \rho_i, i = 1, \cdots, n$. Then $\rho(P, Q)$ turns out to be a Hilbert distance if and only if it is of the form $(C - f(P, Q))^{\frac{1}{2}}, f(P, Q)$ being any positive function in $\mathfrak{C}$ ($C$ is the constant value of $f(\phi, \phi)$).

The representation theory of $\mathfrak{C}$ in $\mathcal{S}$ following Bochner, J. v. Neumann and H. Weyl is carefully gone into, and explicit forms for $\rho^2(P, Q)$-s orthogonal expansions in representation functions of $\mathfrak{C}$ in $\mathcal{S}$ are obtained. They turn out to be absolutely and uniformly convergent expressions of the absolute value square sum type.


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