Kummer [J. Reine Angew. Math. 32, 341–359 (1846)] studied the theory of the periods in cyclotomy and was led to consider the counterpart of the Gauss sum in the cubic case, namely

\[ x = 1 + 2 \sum_{\nu=1}^{(p-1)/2} \cos \left( \frac{2\pi \nu^3}{p} \right) \]

where \( p \) is a prime of the form \( 6n + 1 \). This sum and the two others which extend over the two kinds of cubic non-residues modulo \( p \) are the three roots of the equation

\[ x^3 - 3px - pA = 0, \]

where \( 4p = A^2 + 27B^2, A \equiv 1 \pmod{3} \). An unsolved problem is that of deciding in advance for a given \( p \) whether the root (1) is the largest, middle or smallest root of (2). Kummer classified the primes \( p \) into three classes accordingly and conjectured that the frequencies of these classes are 1/2, 1/3 and 1/6 respectively. His calculations based on the first 45 primes \( p = 6n + 1 \) gave densities of .5333, .3111 and .1556.

The present note extends Kummer’s calculations to the primes less than 10,000; in all, 611 primes. The results do not bear out Kummer’s conjecture. The densities obtained are .4452, .3290 and .2258. These “seem to indicate a trend toward randomness”. On the other hand there is room for the conjecture that the ultimate densities are 4/9, 3/9 and 2/9. The calculation was made on the IAS Computer and required about 15 million multiplications.

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References


Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

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