The author calls a compact manifold $M$ a “Γ-manifold” if it is possible to assign to each ordered pair $(x, y)$ of points of $M$ a third point $z$ called the product $x \cdot y$ in such a way that (1) $x \cdot y$ is a continuous function of the pair $(x, y)$ and (2) for a fixed $x_0$ the correspondences $x \to x \cdot x_0$ and $x \to x_0 \cdot x$ are mappings (of $M$ on $M$) of degrees different from 0 (it is clear that these degrees do not depend on the choice of $x_0$). Every group manifold is of course a Γ-manifold. Furthermore, any sphere $S_n$ of an odd dimension is easily proved to be a Γ-manifold (while no sphere of an even dimension is a Γ-manifold).

The main purpose of the present paper is the proof of the following striking result: If $M$ is a Γ-manifold the homology ring of $M$ (with rational numbers as coefficients) is isomorphic to the homology ring of the product space of spheres $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k}$, where the $n_i$ are odd numbers. Consequently the Betti numbers of $M$ are the coefficients of the polynomial $\prod_{i=1}^k (1 + z^{n_i})$. In other words, the polynomial $\prod_{i=1}^k (1 + z^{n_i})$ is the Poincaré polynomial of the manifold $M$.

This theorem is not only a far-reaching generalization of the well-known results of Pontrjagin about Lie groups, but in addition it throws an entirely new light on these results by emphasizing their elementary topological nature and their independence of analytic or algebraic assumptions (not even the associative law of multiplication is needed). The proof is surprisingly simple and is based upon the known fact that a continuous mapping of a compact manifold $M$ into another compact manifold $M'$ induces a homomorphism of the homology ring of $M'$ into the homology ring of $M$ (this is the so-called “Hopf’s Umkehrhomomorphismus”). Now the product operation defined in a Γ-manifold $M$ can obviously be regarded as a mapping of the product-space $M \times M$ into $M$; hence we have a homomorphism $\Phi$ of the homology ring of $M$ into that of $M \times M$. The latter homology ring is of course determined by the first, and from the assumption (2) about the product operation one derives certain algebraic properties of the homomorphism $\Phi$ which as the author shows cannot be fulfilled unless the homology ring of $M$ has the structure specified above.

[The reviewer wishes to remark that, by using the technique of cohomologies (instead of homologies), it would be easy to prove the author’s result by essentially the same methods for the more general case of compact spaces $M$ which are not necessarily manifolds, assuming that the cohomology ring of $M$ has a finite basis and that $M$ admits a continuous product-operation satisfying the following condition: (2’) For a fixed point $x_0$ the endomorphisms of the cohomology ring of $M$ induced by the transformations $x \to x \cdot x_0$ and $x \to x_0 \cdot x$ are automorphisms; if $M$ is a manifold this condition is easily shown to be equivalent to the previous condition (2).]

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