Strong approximation theorems for density dependent Markov chains.


The author considers the wide variety of density-dependent Markov population models which can be expressed as the solutions of stochastic equations of the form (1) \( X_N(t) = x_0 + \sum_j N^{-1} jY_j(N \int_0^t f_j(X_N(s)) \,ds) \), where \( NX_N(t) \) and \( j \) belong to \( \mathbb{Z}^d \), the \( Y_j \) are independent Poisson processes and \( N \) is a “large” parameter representing a typical population size (see, for example, an earlier paper by the author [J. Appl. Probability 8 (1971), 344–356; MR0287609]). From the law of large numbers for \( Y_j \), equation (1) is, for large \( N \), a.s. close to the equation \( X(t) = x_0 + \int_0^t \sum_j j\langle f_j(X(s)) \,ds \), and an argument exploiting this idea shows directly that \( X_N(t) = X(t) + \varepsilon_N(t) \), where \( \sup_{0 \leq t \leq T} |\varepsilon_N(t)| = O(N^{-1/2}) \). The process \( X_N(t) \) is then compared on the same probability space with an \( N \)-dependent diffusion process \( Z_N(t) \), obtained as the solution of an equation similar to (1), but with \( B_j \) for \( Y_j \), where the \( B_j \) are independent Brownian motions with unit drift and variance. Again the two equations are compared using a.s. path by path approximations, now based on a theorem of J. Komlós, P. Major and G. Tusnády [Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 32 (1975), 111–131; MR0375412], and it is shown that \( \sup_{0 \leq t \leq T} |X_N(t) - Z_N(t)| \) is typically of order \( \log N/N \). Finally, it is established that \( Z_N(t) = X(t) + N^{-1/2}V(t) + \varepsilon_N'(t) \), where \( \sup_{0 \leq t \leq T} |\varepsilon_N'(t)| = O(N^{-1}) \) and where \( V(t) \) is a suitably constructed version of the usual \( N \)-independent diffusion approximation to \( N^{1/2}(X_N(t) - X(t)) \). This is accomplished by comparing \( Z_N \) and \( V \) through the Itô equations which define them. The conditions imposed on the functions \( f_j \) in the statements of the various theorems are not restrictive, and yet the final conclusion, expressing \( X_N(t) \), \( 0 \leq t \leq T \), as \( X(t) + N^{-1/2}V(t) \), with a path by path error which in practical situations is a.s. \( O(\log N/N) \), represents as precise a description of the diffusion approximation as one could wish to achieve.

Andrew D. Barbour

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