If $f_x$ is an isometric map of the straight line $-\infty < x < \infty$ into Hilbert space, then the distance in Hilbert space between two points $f_x$ and $f_y$ is a function $F(t)$ depending only on the quantity $t = |x - y|$. The paper gives two proofs, one operational and one analytical, for the formula

$$F^2(t) = \int_0^\infty \left(\frac{\sin tu}{u^2}\right)^2 d\gamma(u),$$

where $\gamma(u)$ is a monotone function in $0 \leq u < \infty$ and $\int_1^\infty d\gamma(u)/u^2$ converges. If we replace the straight line by a Euclidean space of any finite dimension $m$, in which case the quantity $t$ is the Euclidean distance $(\sum_{k=1}^m (x_k - y_k)^2)^{\frac{1}{2}}$, then the formula generalizes to

$$F^2(t) = \int_0^\infty \left(1 - \Omega_m(tu)\right)/u^2 d\gamma(u),$$

where $\Omega_m(t)$ is a Bessel function. The generalization is proved only by the second method. Finally it should be noted that, according to a previous result of Schoenberg, the limit $m \to \infty$ in the latter formula leads to $\int_0^\infty ((1 - e^{-t^2u})/u) d\gamma(u)$.  

References

1. P. Lvy, Calcul des Probabilits, Paris, 1925. MR0046593
4. I. J. Schoenberg, Metric spaces and positive definite functions, these Transactions, vol. 44 (1938), pp. 522-536. MR1501980

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.