On the theory of characters of commutative topological groups.


This paper deals with the characters of a commutative topological group $S$, in which an invariant measure, with the usual properties of regular Lebesgue measure, is given. (It is convenient to assume that every single point in $S$ has measure 0.) An elegant, quasi-algebraical method is used: The space $L$ of all complex numerical functions $x(g), g \in S$, with finite $\|x\| = \int x(g) \, dg$ is introduced. $L$ is a normed ring, with $\|x\|$ as norm, and the “convolution” $x * y(g) = \int x(g - h)y(h) \, dh$ as multiplication; $L$ possesses no unit, but it may be extended to a normed ring $R$ with unit $e$ by symbolic adjunction of $e$. The general element of $R$ is then $\lambda e + x$, $\lambda$ a complex number, $x \in S$, with norm $|\lambda| + \|x\|$.

$L$ is a maximal ideal in $R$. The main device of this investigation consists in showing that the other maximum ideals $M$ in $R$ are in a one-to-one correspondence with the (continuous) characters $\chi(g)$ of $S$. This correspondence is defined by

$$\text{Restclass of } \lambda e + x \text{ in } R/M = \lambda + \int x(g)\overline{\chi(g)} \, dg,$$

or equivalently by

$$\chi(h) = \frac{\text{Restclass of } x_h \text{ in } R/M}{\text{Restclass of } x \text{ in } R/M}.$$

(Here $x_h(g) \equiv x(g+h)$, and the above quotient is really independent of $x$.) This procedure permits quick derivations of several important facts; e.g.: (1) All measurable characters of $S$ are also continuous. (2) $\lambda e + x$ possesses a reciprocal (in $R$) if and only if $\lambda \neq 0$ and $\lambda + \int x(g)\overline{\chi(g)} \, dg \neq 0$ for all characters $\chi(g)$ of $S$. (3) $R$ is semisimple. (4) If $g_1 \neq g_2$, then a character with $\chi(g_1) \neq \chi(g_2)$ exists.

J. von Neumann

© Copyright American Mathematical Society 2018