This article is a detailed survey of the study of the Kontsevich-Zorich cocycle and applications to linear flows on translation surfaces.

The primary object of study is the $SL(2, \mathbb{R})$ action on the moduli space $\Omega_g$ of pairs $(X, \omega)$, where $X$ is a compact genus $g$ Riemann surface and $\omega$ a holomorphic 1-form on $X$. Such a pair leads to an atlas of charts from $X$ (away from the zeros of $\omega$) to $\mathbb{C}$ whose transition maps are translations; hence such pairs are referred to as translation surfaces. The $SL(2, \mathbb{R})$-action is given by postcomposition with these translation charts.

Each translation surface gives a family (parameterized by $S^1$) of dynamical systems known as linear flows. The flow in direction $\theta \in S^1$ is given by moving at unit speed along leaves of the foliation $\text{Re}(e^{i\theta}\omega) = 0$.

The $SL(2, \mathbb{R})$-action serves as a renormalization dynamics for these linear flows. The Kontsevich-Zorich cocycle is a cocycle over this action defined by considering the bundle over $\Omega_g$ whose fibers are the absolute homology of the surface $X$ (with real coefficients). The growth of this cocycle under the Teichmüller geodesic flow (the action of the diagonal subgroup of $SL(2, \mathbb{R})$) yields strong qualitative and quantitative information (mixing properties, deviation of ergodic averages) on these linear flows.

This article is suitable for advanced graduate students or researchers interested in a detailed survey of the field. The relevant background is provided, and after reviewing much of the development of this field in the 1980’s and 1990’s (much of it due to Veech, Masur, and Zorich) in sections 1–5, the second part of the paper (sections 6–9) discusses state-of-the-art (as of the writing of the paper; the field is growing explosively) examples where the growth of the cocycle is as degenerate as possible, and the ergodic implications thereof.

Jayadev S. Athreya

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