Integration of a holomorphic 1-form \( \omega \) on a Riemann surface \( X \) produces a local coordinate away from the zeros of \( \omega \) which forms part of a flat structure: any two such coordinates differ by a translation. Thus, a translation surface is a triple \((X, \omega, P)\) where \( P \subseteq X \) is a finite set (“marked points”) containing the zeros of \( \omega \). The set \( \text{Hol}(X, \omega, P) \) of holonomy vectors, the integrals of \( \omega \) along geodesics in \( X \setminus P \) joining points of \( P \), is a discrete subset of \( \mathbb{R}^2 \). Many number-theoretical results for \( \mathbb{Z}^2 \) can be generalized to \( \text{Hol}(X, \omega, P) \); this paper is particularly concerned with the e-Minkowski constant (“e” for “elliptic”) for Veech surfaces.

The so-called “Minkowski’s First Theorem” states that if \( C \subseteq \mathbb{R}^2 \) is convex and symmetric with respect to the origin and contains no points of \( \mathbb{Z}^2 \setminus \{0\} \), then \( \text{area}(C) < 4 \). Motivated by this, the Minkowski constant \( M(X, \omega, P) \) of a translation surface is defined to be \( \frac{1}{4} \sup \text{area}(C) \), where \( C \) varies over the bounded, convex regions symmetric with respect to the origin and disjoint from the holonomy vectors. The e-Minkowski constant \( M^e(X, \omega, P) \) is obtained by requiring \( C \) also to be the interior of an ellipse.

The Veech group \( \Gamma(X, \omega, P) \subseteq \text{SL}(2, \mathbb{R}) \) is the collection of differentials of the affine (with respect to \( \omega \)) automorphisms of \( X \). A Veech surface (or lattice surface) is a translation surface for which the Veech group is a lattice. The main result is that for Veech surfaces, \( M^e(X, \omega, P) \) is equal to \( \frac{1}{4} \) the maximum area of the “strong support ellipses”, which are ellipses centered at the origin with boundary meeting \( \text{Hol}(X, \omega, P) \) in at least 6 points, but having no interior holonomy vectors.

The elements of the proof of this result are used to give an algorithm for calculating the e-Minkowski constant \( M^e \). This involves an algorithm for drawing the spine of the translation surface, which is a subset of \( \mathbb{R}^2 \) obtained by moving the shortest holonomy vectors via the natural action of \( \text{SL}(2, \mathbb{R}) \) on translation structures, and it is a tree whose edges are hyperbolic geodesics and which is invariant under the action of \( \Gamma(X, \omega, P) \subseteq \text{SL}(2, \mathbb{R}) \). A series of subsidiary results are required to finitize those steps of the spine algorithm which require taking maxima or minima over \( \text{Hol}(X, \omega, P) \).

Some examples of \( M^e \) are calculated, including the flat torus marked by a single point, and translation surfaces obtained by identifying parallel edges of a regular \( n \)-gon (\( n \) even) or two copies of the \( n \)-gon (\( n \) odd), \( n \geq 5 \). Most of the corresponding Veech groups are non-arithmetic.

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References

and P. Sarnak; American Mathematical Society, Providence, RI, 1997) 165–189. MR1429199

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

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