A Diophantine duality applied to the KAM and Nekhoroshev theorems.
(English summary)

The authors of this paper obtain two stability results in the model case of a perturbation of a constant vector field on a torus. They develop a new approach to the perturbation theory of quasiperiodic flows using periodic approximations instead of the classical small divisors approach. This periodic approximation idea goes back to [P. Lochak, Uspekhi Mat. Nauk 47 (1992), no. 6(288), 59–140; MR1209145]. It was further developed by L. Niederman [Ergodic Theory Dynam. Systems 27 (2007), no. 3, 905–928; MR2322185] and by A. Bounemoura and Niederman [Ann. Inst. Fourier (Grenoble) 62 (2012), no. 1, 277–324; MR2986272]. The new approach is based on two dual Diophantine problems, which the authors relate using the geometry of numbers.

The situation is that of a constant vector field $X_\alpha = \alpha$ for a given $\alpha \in \mathbb{R}^n \setminus \{0\}$ on the $n$-torus $\mathbb{T}^n$. Let $m$ be the number of independent rational relations among the components of $\alpha$. Then there is an invariant foliation on $\mathbb{T}^n$ whose leaves are diffeomorphic to $\mathbb{T}^{n-m}$ and whose leaf space is diffeomorphic to $\mathbb{T}^m$. Let $d = n - m$.

The first stability result is of Nekhoroshev type. It deals with adding a small analytic perturbation $P$ to $X_\alpha$ with no restrictions on $\alpha$ (and hence no restrictions on $d$). It gives an analytic conjugacy of $X_\alpha + P$ with a “partial” normal form $X_\alpha + N + R$ where $X_\alpha + N$ is a normal form where $N$ commutes with $X_\alpha$, and $R$ is a small remainder. This implies a stability in finite time on an appropriate time scale by comparing the flow of $X_\alpha + P$ with the simpler flow of $X_\alpha + N$.

The second stability result is of KAM type. It is only valid when $d = n$ and $\alpha$ satisfies the Bruno-Rüssmann condition. It gives an analytic conjugacy of $X_\alpha$ with $X_\alpha + X_\beta + P$ where $X_\beta$ is a constant vector field depending on $P$. Because $d = n$, the constant vector fields are precisely the vector fields which commute with $X_\alpha$, and so this result states that the existence of a vector field $N = X_\beta$ that commutes with $X_\alpha$ such that $X_\alpha$ and $X_\alpha + N + P$ are analytically conjugated. The authors call this an “inverted” normal form. The analytic conjugacy implies a stability in infinite time, not for the original perturbed vector field $X_\alpha + P$, but for $X_\alpha + N + P$. This is precisely the classical KAM theorem for constant vector field proved by Arnold and Moser.

References

30. Waldschmidt, M.: Topologie des points rationnels, Cours de Troisième Cy-

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.