The paper under review considers the following linear eigenvalue problem:
\[
\begin{aligned}
&-D\Delta\phi - 2\alpha\nabla m \cdot \nabla \phi + V\phi = \lambda\phi \quad \text{in } \Omega, \\
&\partial_n\phi = 0 \quad \text{on } \partial\Omega,
\end{aligned}
\]
where \(\Omega\) is a bounded domain in \(\mathbb{R}^N\) with smooth boundary \(\partial\Omega\); the constants \(D > 0\) and \(\alpha\) are diffusion and advection coefficients, respectively; \(m(x)\) and \(V(x)\) are given smooth functions on \(\Omega\), \(\partial_n\phi = n(x) \cdot \nabla \phi(x)\), and \(n(x)\) is the unit exterior normal to \(\partial\Omega\) at \(x\). The case \(\alpha = 0\) of this problem is well studied, and this paper considers the case \(\alpha > 0\) since the case \(\alpha < 0\) can be transformed into this case by replacing \(\alpha\) and \(m\) with \(-\alpha\) and \(-m\), respectively. If we multiply both sides of the equation by \(e^{2(\alpha/D)m}\), then we get the equation in divergence form:
\[
\begin{aligned}
-D\nabla \cdot \left[ e^{2(\alpha/D)m} \nabla \phi \right] + e^{2(\alpha/D)m} V\phi = \lambda e^{2(\alpha/D)m}\phi \quad \text{in } \Omega, \\
\partial_n\phi = 0 \quad \text{on } \partial\Omega.
\end{aligned}
\]
Then the smallest eigenvalue \(\lambda(\alpha, D)\) of the problem is characterized by
\[
\lambda(\alpha, D) = \inf_{\phi \in W^{1,2}(\Omega), \phi \neq 0} \frac{\int_\Omega e^{2\alpha m/D} (|\nabla \phi|^2 + V\phi^2) dx}{\int_\Omega e^{2\alpha m/D} \phi^2 dx}.
\]
This implies that
\[
\min_{\Omega} V \leq \lambda(\alpha, D) \leq \max_{\Omega} V
\]
for any \(\alpha, D\). It is important to determine the asymptotic behavior of \(\lambda(\alpha, D)\) for sufficiently large or small \(\alpha, D\).

The authors of the paper under review prove the following results.

1. If all critical points of \(m(x)\) are nondegenerate, then
\[
\lim_{\alpha/((1+D)\ln(2+D)) \to \infty} \lambda(\alpha, D) = \min_{x \in \mathcal{M}} V(x),
\]
where \(\mathcal{M}\) is the set of points of local maximum of \(m\).

2. Let \(\alpha\) be a fixed positive constant. If \(|\nabla m| \neq 0\) on \(\partial\Omega\), then
\[
\lim_{D \to 0} \lambda(\alpha, D) = \min_{x \in \Sigma_1 \cup \Sigma_2} \left\{ V(x) + \alpha \sum_{i=1}^N \left( |\kappa_i(x)| + \kappa_i(x) \right) \right\},
\]
where
\[
\Sigma_1 := \{ x \in \Omega : |\nabla m(x)| = 0 \},
\]
\[
\Sigma_2 := \{ x \in \partial\Omega : |\nabla m(x)| = n(x) \cdot \nabla m(x) > 0 \}.
\]
Here, when \(x \in \Omega\), \(\kappa_1(x), \ldots, \kappa_N(x)\) are eigenvalues of \(D^2m(x)\); when \(x \in \partial\Omega\), \(\kappa_1(x), \ldots, \kappa_N(x)\), \(\kappa_N(x) = 0\) are eigenvalues of \(D^2m_{\partial\Omega}(x)\), where \(m_{\partial\Omega}(x)\) is the restriction of \(m(x)\) on \(\partial\Omega\).
For each $k \in [0, \infty)$,
\[
\lim_{(\alpha,D) \to (0,0),\alpha^2/D \to k} \lambda(\alpha, D) = \min \left\{ \min_{\Sigma_3} \{ k|\nabla \partial \Omega m|^2 + V \}, \min_{\Omega} \{ k|\nabla m|^2 + V \} \right\},
\]
where $\nabla \partial \Omega = \nabla - \mathbf{n}(\mathbf{n} \cdot \nabla)$ and $\Sigma_3 := \{ x \in \partial \Omega : \partial_n m(x) > 0 \}$. Particularly,
\[
\lim_{\alpha \to 0} \inf_{D > 0} \lambda(\alpha, D) = \min_{\Omega} V.
\]

Let $\Sigma_1$ and $\Sigma_2$ be as in (2). Then
\[
\lim_{(\alpha,D) \to (0,0),\alpha^2/D \to \infty} \lambda(\alpha, D) = \min_{\Sigma_1 \cup \Sigma_2} V.
\]

The authors conjecture that, provided $\min_{\Sigma_1 \cup \Sigma_2} V > \min_{\Omega} V$, for small $\alpha$, there is a unique $D_\ast = D_\ast(\alpha) > 0$ such that $\lambda(\alpha, D)$ is monotone decreasing for $D \in (0, D_\ast)$ and monotone increasing for $D \in (D_\ast, \infty)$. In the last section they show how these asymptotic behaviors of $\lambda(\alpha, D)$ yield new insights into the persistence of a sinking phytoplankton species in poorly mixed water columns.

References


Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

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