A conjecture attributed to Arnol’d asserts that every closed exact Lagrangian submanifold $L$ in the cotangent bundle of a closed manifold $N$ should be Hamiltonian isotopic to the zero section. This paper represents one of the final steps in proving a weaker version of this conjecture, namely that the inclusion of $L$ into $T^*N$ is a homotopy equivalence: specifically, the present paper proves this statement provided that $L$ has vanishing Maslov class, and later work by T. Kragh and the author [Geom. Topol. 17 (2013), no. 2, 639–731; MR3070514] proves that the Maslov class condition is automatically satisfied.

Many aspects of the basic program used to establish the main result go back to [K. Fukaya, P. Seidel and I. Smith, in *Homological mirror symmetry*, 1–26, Lecture Notes in Phys., 757, Springer, Berlin, 2009; MR2596633] and references therein, where it was proven that, at least if $L$ is simply connected and spin, it is isomorphic to the zero section in the derived Fukaya category over any given field, and hence in particular has the same cohomology as $N$ over any field. In fact, the author shows in an appendix at the end of this paper that, when combined with more recent developments about the wrapped Fukaya categories of cotangent bundles, the Fukaya-Seidel-Smith approach can be used to show that, even without assuming $L$ to be simply connected or spin, the inclusion $L \to T^*N$ induces an isomorphism on cohomology. So, by the Whitehead and Hurewicz theorems, the main result will follow as soon as one shows that $L \to T^*N$ induces an isomorphism on $\pi_1$.

The main tool constructed in this paper is a version $S(T^*N)$ of the wrapped Fukaya category whose objects are exact Lagrangian submanifolds of $T^*N$ equipped with local systems of chain complexes. The Floer complex $CW^*(E^0, E^1)$ associated to two objects $(E^0, L^0)$ and $(E^1, L^1)$ of this category (where $E^i$ is a local system of chain complexes over $L^i$) is given, as an ungraded vector space, by the direct sum $\bigoplus_x \text{Hom}(E^0_{x(0)}, E^1_{x(1)})$, where the sum is over Hamiltonian chords $x$ from $L^0$ to $L^1$ for a Hamiltonian which is quadratic at infinity. A natural combination of the usual holomorphic polygon counts with the parallel transport maps associated to the local systems makes this extended Fukaya category into an $A_\infty$-category. A fairly deep extension of the main result of [M. Abouzaid, Adv. Math. 228 (2011), no. 2, 894–939; MR2822213] shows that trivial local systems over a cotangent fiber $T^n N$ split-generate $S(T^*N)$.

By considering the action of $HW^0(T^*_n N) \cong \mathbb{Z}\pi_1(N)$ on $HW^*(T^*_n N, E)$ for a general object $(E, L)$ of $S(M)$, one associates to $(E, L)$ local systems over the zero section of $T^*N$ having fiber $HW^k(T^*_n N, E)$, and under suitable hypotheses $E$ is shown to be isomorphic in $S(T^*N)$ to an iterated cone on these local systems. From this it follows that when $N$ is simply connected $E$ is itself a trivial local system; from the arbitrariness of $E$ it follows that $L$ must be, like $N$, simply connected. The case when $N$ is not simply connected requires the formulation of a similar Fukaya category associated to a covering space, which effectively allows one to reduce to the simply connected case (though the noncompactness of the universal cover requires this to be done with some care).

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References

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

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