Local rigidity of homogeneous parabolic actions: I. A model case. (English summary)


The paper investigates a weak version of local differential rigidity for specific higher rank abelian unipotent (or parabolic) actions. A typical example of such action is given by the rank 2 abelian $\mathbb{R}^2$ action $\alpha$ generated by the two upper unipotents in $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ acting on $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})/\Gamma$, where $\Gamma \subseteq \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ is a co-compact lattice:

$$
\alpha(t_1, t_2) = \begin{pmatrix} 1 & t_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (t_1, t_2) \in \mathbb{R}^2.
$$

Other examples of parabolic actions are presented in the article.

This contribution further develops the KAM-iteration technique introduced by the authors in a previous paper [Ann. of Math. (2) 172 (2010), no. 3, 1805–1858; MR2726100], in which they proved local differential rigidity for higher rank partially hyperbolic actions on tori, and greatly enriches the universe of higher rank abelian actions that exhibit rigidity phenomena.

Note that unipotent actions are not structurally stable. Indeed, the candidate for conjugacy to the initial map is not unique, but rather belongs to a parametric family. As the authors observe, this phenomenon appears for other classification problems, for example, the classification of circle diffeomorphisms. In that case the natural parameter is given by the rotation number.

We present a summary of the proof, following the authors’ introduction:

Step 1. The conjugacy equation is linearized at the unperturbed action and a suitable operator introduced. Solving the linearized equation is shown to be equivalent to finding an inverse of the operator on a proper subspace of data.

Step 2. The obstructions for solving the linearized conjugacy equation are found. There are infinitely many obstructions, in contrast to the standard KAM method for translations where the obstruction is unique. Those obstructions are invariant distributions and can be described using harmonic analysis.

Step 3. All but finitely many obstructions vanish due to the commutation relations. This phenomenon has been observed before, for example in the case of hyperbolic actions, and it is called the higher-rank trick.

Step 4. The remaining finitely many parameters are absorbed by allowing for a group automorphism, or a standard perturbation, or by adjusting parameters. Accordingly, one obtains a different notion of rigidity.

Step 5. (a) The linearized equation is solved. This involves the glueing of solutions constructed in certain invariant subspaces.

(b) Tame estimates are obtained for the solution. Tameness refers to finite loss of regularity in the chosen collection of norms in the Fréchet spaces, such as $C^r$ or Sobolev norms.

Step 6. The perturbation can be split into two terms due to the commutation relations:
one for which the linearized equations are satisfied, and the other “quadratically small” with tame estimates.

Step 7. The conjugacy provided by the solution of the linearized equation transforms the perturbed action into an action quadratically close to the target with a fixed loss of regularity in the estimates.

Step 8. The process is iterated and the product of intermediate conjugacies converges to a conjugacy between the (modified) original and perturbed action (with adjusted parameters if necessary).

The final result is that for a two-parameter family of sufficiently small perturbations of the action \(\alpha\), satisfying certain transversality conditions, there exists a parameter for which the perturbation is smoothly conjugate to the action up to an automorphism of the acting group. This weak form of rigidity for the parabolic action in question is optimal.


References


15. F. Ramirez, Coycles over higher-rank abelian actions on quotients of semisimple Lie groups, J. Mod. Dyn., 3 (2009), 335–357. MR2538472


Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.