A geometric criterion for generating the Fukaya category. (English summary)


A full subcategory $\mathcal{B}$ of an $A_\infty$-category $\mathcal{C}$ split-generates if every object of $\mathcal{C}$ is isomorphic, in the derived category $D\mathcal{C}$, to a direct summand in a twisted complex formed from the objects of $\mathcal{B}$. The article under review establishes a beautiful new criterion for split-generation of Fukaya categories. Let $M$ be a Liouville domain, i.e., a compact, exact symplectic manifold with contact-type boundary, and $\mathcal{W}(M)$ its wrapped Fukaya category [M. Abouzaid and P. Seidel, Geom. Topol. 14 (2010), no. 2, 627–718; MR2602848]. A collection of objects, forming a full subcategory $\mathcal{B}$ of $\mathcal{W}(M)$, define an ‘open-closed string map’

$$HH_*(\mathcal{B}) \to SH^*(M),$$

from the Hochschild homology of the subcategory to the symplectic cohomology ring of $M$. The criterion is as follows: if the unit $1 \in SH^*(M)$ lies in the image of the open-closed string map then $\mathcal{B}$ split-generates $\mathcal{W}(M)$.

Geometrically, if this condition holds then any point $x \in M$ lies on a pseudoholomorphic polygon bounding Hamiltonian images of the Lagrangian branes in $\mathcal{B}$; however, the statement is a good deal more precise than that. Abouzaid’s argument claims, when the open-closed string map hits 1, that for each object $K$ of $\mathcal{W}(M)$, the identity morphism $1_K \in \text{Hom}_{\mathcal{W}(M)/\mathcal{B}}(K, K)$ in the quotient $A_\infty$-category [V. V. Lyubashenko and S. Ovsienko, Homology Homotopy Appl. 8 (2006), no. 2, 157–203; MR2259271] is equal to zero. This signifies that $\mathcal{B}$ split-generates $\mathcal{W}(M)$. The claim that $1_K = 0$ is proved using a version of the Cardy relation from open-closed TQFT.

The argument is robust enough, in principle, to be adaptable to the case of Fukaya categories for compact symplectic manifolds, with the open-closed string map taking quantum cohomology as its target. Indeed, the author has announced forthcoming work joint with K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono which will prove the generation criterion in that context. In both non-compact and compact versions, Abouzaid’s criterion promises to influence research on Fukaya categories, and in particular on homological mirror symmetry, in a decisive way. N. Sheridan has led the way with his recent paper [“Homological mirror symmetry for Calabi-Yau hypersurfaces in projective space”, preprint, arXiv:1111.0632], which invokes the compact version of the criterion.

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References


Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

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