Abouzaid, Mohammed (1-MIT); Smith, Ivan (4-CAMB-CM)

Homological mirror symmetry for the 4-torus. (English summary)


M. Kontsevich’s wonderful mirror symmetry conjecture [in Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), 120–139, Birkhäuser, Basel, 1995; MR1403918] predicts that the bounded derived category of coherent sheaves on a Calabi-Yau manifold should be equivalent to the derived Fukaya category of (appropriately decorated) Lagrangian submanifolds of another Calabi-Yau—the mirror manifold. In fact one should take split-closure on the Fukaya side, and one can get a slightly stronger conjecture by using a dg-enhancement on the coherent sheaf side. In an amazing tour-de-force P. Seidel has proved the conjecture for quartic $K3$ surfaces [“Homological mirror symmetry for the quartic surface”, preprint, arxiv.org/abs/math/0310414]. The aim of the paper under review is to deal with the other Calabi-Yaus of complex dimension 2: abelian surfaces.

Since the conjecture was also proved early on for elliptic curves $E$ [A. Polishchuk and E. Zaslow, Adv. Theor. Math. Phys. 2 (1998), no. 2, 443–470; MR1633036] one might expect to be able to deduce it for the abelian surface $E \times E$. The idea is that objects of the derived category of $E \times E$ should correspond via the Fourier-Mukai construction to functors from $D(E)$ to itself. Dually objects of the Fukaya category of $E \times E$ should correspond to functors from the Fukaya category of $E$ to itself.

The problem is that neither is strictly true, and difficult current technology is only just beginning to get a grip on solving these problems. So at this point the nonexpert should stop reading, appreciate the wonderful result and wait for the theory to settle down and simplify. The bulk of the paper is an account of the fearsome technicalities needed to overcome these problems.

On the coherent sheaf side one has to pass to a dg enhancement, re-prove homological mirror symmetry for the enhanced derived category of $E$, and then use B. Toën’s recent (difficult) result [Invent. Math. 167 (2007), no. 3, 615–667; MR2276263] that the category of endofunctors is indeed the dg enhancement of the derived category of the product.

The main content of this paper is a discussion of the symplectic, Fukaya category side. Here the modifications are a little more serious, using the Lagrangian correspondences and pseudoholomorphic quilts of S. Mau, K. Wehrheim and C. T. Woodward [see, e.g., K. Wehrheim and C. T. Woodward, Quantum Topol. 1 (2010), no. 2, 129–170; MR2657646]. For Lagrangians on the product which are cones of products of Lagrangians in the two factors this gives endofunctors as required. For more general Lagrangians the theory is more complicated. So the authors find an ingenious way of making do with such product Lagrangians plus some Hochschild cohomology information that takes care of the all-important diagonal Lagrangian representing the identity functor. (More conceptual arguments are now available; see [M. Abouzaid, “A geometric criterion for generating the Fukaya category”, preprint, arxiv.org/abs/1001.4593, Publ. Math. Inst. Hautes Études Sci., to appear].)

The upshot is that product Lagrangians made from only two Lagrangians on the elliptic curve split generate, and that the category of Lagrangian correspondences really becomes equivalent to the Fukaya category of the product once we pass to twisted
complexes and take the split closure.

The proofs are often brief and assume familiarity with diverse and tricky arguments from both sides of mirror symmetry: Floer homology, homological and $A_\infty$ algebra, coherent sheaves, semi-homogeneous sheaves on abelian varieties, Hochschild cohomology, etc. This is probably necessary—the paper is 66 pages long as it is—but it is forbidding for non-experts. Hence some arguments are rather discursive, explained as “by the methods of another 66 page paper applied slightly differently here, . . .”. Also, the section on quilts (a yet-to-be fully-developed theory) relies on axioms that the authors expect the final theory to satisfy.

The methods of the paper illustrate the enormous power of taking generators under split closure. In this way the authors never need to augment their Lagrangians with flat connections, and, remarkably, they may even remove the need for the co-isotropic branes of A. Kapustin and D. O. Orlov [J. Geom. Phys. 48 (2003), no. 1, 84–99; MR2006226]. The authors also describe some nice applications, for instance giving structure results for genus 2 Lagrangian submanifolds of the 4-torus.

Richard P. Thomas

References


7. S. DONALDSON and I. SMITH, Lefschetz pencils and the canonical class for symplectic four-manifolds, Topology 42 (2003), 743 – 785. MR 1958528 405 MR1958528


14. ———, Metamorphosis of holomorphic polygons under Lagrange surgery, preprint, 2008. 376, 387, 390, 393, 404


38. ———, Abelian variety and spin representation, unpublished manuscript, 1998. 430


46. ———, Extensions of homogeneous coordinate rings to $A_{\infty}$-algebras, Homology Homotopy Appl. 5 (2003), 407 – 421. MR 2072342 407 MR2072342


Abouzaid Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139-4307, USA; abouzaid@math.mit.edu Smith Centre for Mathematical Sciences, University of Cambridge, Cambridge CB3 0WB, United Kingdom; is200@cam.ac.uk

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