In this paper, the theory of smoothness of a conjugacy of a circle diffeomorphism with a rotation is revisited. Of special interest is the case of minimal smoothness ($C^{2+\alpha}$-diffeomorphism) ($\alpha > 0$) which was considered by Y. G. Sinaï and K. M. Khanin in [Uspekhi Mat. Nauk 44 (1989), no. 1(265), 57–82, 247; MR0997684], and Y. Katznelson and D. S. Ornstein in [Ergodic Theory Dynam. Systems 9 (1989), no. 4, 643–680; MR1036902]. Until recently, the following result by Katznelson and Ornstein in [op. cit.] has been regarded as a crown of this theory: if $f$ is a $C^{2+\alpha}$-smooth orientation-preserving circle diffeomorphism with rotation number $\rho$ belonging to the Diophantine class $D_\delta$, where $0 < \delta < \alpha \leq 1$, then the conjugacy is $C^{1+\alpha-\delta-\varepsilon}$-smooth for an arbitrary small positive $\varepsilon$. (We recall that an irrational number $\rho$ is said to belong to the Diophantine class $D_\delta$, where $\delta \geq 0$, if there exists a $C > 0$ such that $|\rho - \frac{p}{q}| \geq Cq^{-2-\delta}$ for any rational number $p/q$.)

In the present paper, the authors consider again the case of minimal smoothness ($C^{2+\alpha}$-diffeomorphisms). They prove the following result (which was first announced in [Y. G. Sinaï and K. M. Khanin, op. cit.]):

Any $C^{2+\alpha}$-smooth orientation-preserving circle diffeomorphism $f$ with rotation number $\rho \in D_\delta$, where $0 < \delta < \alpha \leq 1$, is $C^{1+\alpha-\delta}$-smooth conjugate to the rotation of the circle by angle $\rho$. This result is stronger than the result of Katznelson and Ornstein above, and gives the first sharp estimate for the smoothness of the conjugacy of a $C^{2+\alpha}$-diffeomorphism with a rotation; exponents higher than $\alpha - \delta$ cannot be reached (as demonstrated by examples constructed in [Y. Katznelson and D. S. Ornstein, op. cit.]). In particular, it implies that the $f$-invariant measure is absolutely continuous, and the density is Hölder continuous with the exponent $\alpha - \delta$.

The approach used is conceptually new, and is based on cross-ratio distortion estimates which make the proof elementary.

References


Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

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