Parabolic geometries. I.
Background and general theory.
Mathematical Surveys and Monographs, 154.

An excellent book, serving both as a timely introduction to parabolic geometry and as a general introductory work for Lie groups and Cartan geometries.

The first two chapters can be read as stand-alone. The first covers Cartan geometries in exhaustive detail, building up from basic principles in differential geometry. In an elegant fashion, it sets out all theory and objects needed—homogeneous spaces, principal bundles, connections—to define and understand Cartan geometries.

The second chapter covers the theory of semisimple Lie algebras and Lie groups. Again, starting from basic principles, the authors prove the important fundamental theorems in the subject, such as the classification of simple Lie groups and their representations, and define such concepts as Verma modules, infinitesimal character and the Casimir element. The whole chapter would not be out of place in a textbook on Lie group theory; its treatment of real Lie groups and real representations is particularly notable.

Parabolic geometries are a method for bringing general results to bear on a whole host of disparate geometries (such as conformal, projective, almost Grassmannian, almost quaternionic, CR, path geometries, etc.) under one consistent framework with universal results. The work of describing this takes up the last three chapters of the book. All of these geometries can be construed as the curved analogues of the homogeneous space $G/P$, with $G$ semi-simple and $P$ parabolic. They are described by a principal $P$-bundle over the manifold, and a one-form on this bundle with values in $\mathfrak{g}$, the Lie algebra of $G$ with certain natural properties. The data is in a specific sense uniquely described by the underlying geometry, and the book presents the concepts—the grading $\mathfrak{g}$, the Kostant co-differential $\partial^*$ with its associated homology and co-homology, regularity and normality of curvature—needed to make sense of this. Examples are described and analysed in detail in the fourth chapter, a useful way to connect the general theory with known geometries. The fifth chapter concludes with a presentation of classes of objects belonging to the underlying geometry and manifold—Weyl connections and distinguished curves—and shows how these derive from the general theory.

This review cannot do justice to the power and generality of parabolic geometry theory, but this book certainly does.

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