On the quantum homology algebra of toric Fano manifolds. (English summary)


A number of recent advances in symplectic topology, notably the constructions of Calabi quasi-morphisms on certain symplectic manifolds beginning in [M. Entov and L. Polterovich, Int. Math. Res. Not. 2003, no. 30, 1635–1676; MR1979584], have shown that interesting consequences follow when the quantum cohomology of a symplectic manifold has special algebraic structure. In particular, Entov and Polterovich’s work initially required the degree-zero quantum cohomology of the manifold to be semisimple as an algebra (i.e. to have no nilpotent elements, or equivalently, to split as a direct sum of fields); McDuff later observed that their constructions continue to work under the weaker assumption that the degree-zero quantum cohomology algebra contains a field as a direct summand. Either of these conditions, especially semisimplicity, is fairly strong, so the question naturally arises as to just which symplectic manifolds satisfy them. In particular, Entov and Polterovich showed [in Toric topology, 47–70, Contemp. Math., 460, Amer. Math. Soc., Providence, RI, 2008; MR2428348] that all toric Fano symplectic manifolds of complex dimension 2 have semisimple quantum cohomology, and they asked whether the same always holds for toric Fano manifolds of arbitrary dimension.

The present paper develops a new set of tools, motivated by mirror symmetry, to address the structure of the quantum cohomology of toric Fano manifolds, and uses these tools to answer Entov and Polterovich’s question negatively. To a toric Fano variety $M$ one can associate a Landau-Ginzburg superpotential $W$ (which can be expressed as a Laurent polynomial $W: (\mathbb{C}^*)^n \to \mathbb{C}$), and a theorem of A. B. Givental [in Topological field theory, primitive forms and related topics (Kyoto, 1996), 141–175, Progr. Math., 160, Birkhäuser Boston, Boston, MA, 1998; MR1653024] is seen to imply that the quantum cohomology $QH^*(M, \omega)$ of $X_{\Sigma}$ is isomorphic to the Jacobian ring of $W$. From this, the authors deduce that the relevant algebraic properties of $QH^0(M, \omega)$ can be read off from the behavior of $W$ in a very direct way: $QH^0(M, \omega)$ is semisimple provided that all critical points of $W$ are nondegenerate, and $QH^0(M, \omega)$ has a field as a direct summand provided that at least one critical point is nondegenerate.

Toric Fano manifolds of complex dimension at most four have been classified and tabulated in sufficient detail as to make it feasible for a computer to examine the critical points of their respective Landau-Ginzburg superpotentials, and the authors find that, out of the 124 toric Fano 4-folds, three have the property that, when one chooses the monotone toric symplectic form, the Landau-Ginzburg superpotential has a degenerate critical point. Thus these three toric Fano 4-folds provide negative answers to Entov and Polterovich’s question. More positively, the computer search also reveals that all of the Landau-Ginzburg superpotentials of toric Fano manifolds of complex dimension at most four have at least one nondegenerate critical point; thus the authors’ result shows that these manifolds’ associated quantum cohomology (with any toric symplectic form) has a field direct summand, implying the existence of a Calabi quasimorphism by McDuff’s observation. It should also be mentioned that, without any constraint on the dimension, a toric Fano manifold $X$ has the property that, for generic toric symplectic forms, the quantum cohomology is semisimple. The authors include a proof
of this result, which was also proven by H. Iritani [J. Reine Angew. Math. 610 (2007), 29–69; MR2359850] and by K. Fukaya et al. [“Lagrangian Floer theory on compact toric manifolds. I”, preprint, arxiv.org/abs/0802.1703]. Moreover, it is shown that the monotone symplectic form $\omega_0$ is the “worst” possible choice of toric symplectic form with respect to these properties of $QH^0$: if $\omega$ is some other toric symplectic form and if $QH^0(M, \omega_0)$ is semisimple (resp. contains a field direct summand) then $QH^0(M, \omega)$ also is semisimple (resp. contains a field direct summand).

In a somewhat different direction, the authors produce an example where the Entov-Polterovich quasimorphism is non-unique: namely, if $(X, \omega)$ is a sufficiently small blowup of $\mathbb{C}^2$ at one point, then the Entov-Polterovich construction gives a quasimorphism from a choice of either of two idempotents in $QH^0(M, \omega)$, and the authors evaluate these two quasimorphisms on Hamiltonian loops and find that they disagree.

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References

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

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