Let us start by noting that the overall result of this paper, namely Theorem 1.1, exactly reproduces A. N. Kolmogorov’s celebrated 1954 result \[\text{Dokl. Akad. Nauk SSSR (N.S.) 98 (1954), 527–530; MR0068687}\] on the persistence of Diophantine invariant tori in analytic nondegenerate near integrable Hamiltonian systems (together with the conjugacy of the flow on the torus to a linear flow). This alone may justify some further remarks of a historical character.

For various reasons, not all mathematical, it has long been generally asserted that Kolmogorov’s proof was not complete. This is now recognized to be wrong. Kolmogorov did provide a complete proof, albeit in a rather compact form (four pages). Even the shortness of the original text is by itself quite interesting and may derive from several causes. First one can trace it to a Russian tradition of oral communication, itself influenced in an intricate way by (non-mathematical) history. After all, Kolmogorov’s famous 1941 turbulence paper is prefaced with a moving appeal about the Barbarossa operation, and in 1954 Stalin had been dead for just a year. From a more scientific perspective, the 1954 note followed an equally short note on flows on the torus [A. N. Kolmogorov, \text{Doklady Akad. Nauk SSSR (N.S.) 93 (1953), 763–766; MR0062892}] and one may remark that the authors of the present paper have followed a similar route [cf. K. M. Khanin, J. Lopes Dias and J. Marklof, Comm. Math. Phys. 270 (2007), no. 1, 197–231; MR2276445]. In the 1954 note it can confidently be asserted that Kolmogorov’s genius went first into stating the correct result, which had escaped the efforts of both Poincaré and Weierstrass (among a few others). Technically speaking, Kolmogorov devised an original weak normal form, with two main ideas, namely assuming Siegel’s Diophantine condition and focusing (“zooming in”) on the torus at hand. Given these—remarkable—ideas, the proof becomes in effect quite easy and completely elementary. One should add the fact that Kolmogorov had started his career in the then fashionable (for deep historical reasons) field of hard Fourier analysis, devising in his youth “monstrous” Fourier series with complicated convergence and divergence properties. Compared to his early works, the “harmonic analysis” entering into the proof of the result on invariant tori is essentially trivial. The above may give a flavor of how one can reconstruct the simple fact that Kolmogorov devoted just four pages to that proof. This original proof has been very nicely elucidated (and can be taught at undergraduate level) in a pedagogical paper by G. Benettin et al. [Nuovo Cimento B (11) 79 (1984), no. 2, 201–223; MR0743977]. Even this complete “elucidation” hardly occupies 20 easy-to-read small pages.

Kolmogorov’s result having acquired the doubly wrong reputation of being difficult and with an incomplete proof was subsequently the subject of innumerable variants, improvements, etc. forming what is nowadays called KAM theory, of which the 1954 note still constitutes the core result. Many techniques have been introduced, which in effect are often simple variants or rephrasings of the original ones. For instance, Kolmogorov’s original converging algorithm or variants thereof may (not surprisingly, as Picard and Hadamard already knew) be cast in the form of fixed point results in
various function spaces. But to return to the present paper, at least two proofs are substantially different from the original one. First, one has tried to directly analyse the convergence of the normalizing (Lindstedt) series describing the putative invariant torus. This direct method was first successfully applied by L. H. Eliasson in [Math. Phys. Electron. J. 2 (1996), Paper 4, 33 pp. (electronic); MR1399458], which is a 1996 publication of the original 1988 preprint (see the review of that item by J. Pöschel, in MathSciNet) and was subsequently refined by G. Gallavotti and coworkers among others. Second, as recounted in the introduction to the present paper (to which we refer the reader), it has long been recognized as tantalizing to devise a renormalization scheme in order to (re-)prove Kolmogorov’s result, and this is what the present paper finally achieves. It is important, if perhaps disappointing, to note that both direct methods and renormalization methods are more cumbersome, more intricate and at present (more than half a century later) much less powerful than Kolmogorov’s original scheme or rather immediate variants thereof. So why bother? Essentially because Kolmogorov’s proof does not tell the whole truth. Especially it does not explain or even give a clue as to what should be the optimal arithmetic condition. In the one-dimensional case, this finally was understood [J.-C. Yoccoz, Astérisque No. 231 (1995), 3–88; MR1367353] using both complex dynamics and renormalization-type reasonings. So one of the incentives for developing direct methods and renormalization in the multidimensional case is that they would tell us the secrets of convergence and in particular of arithmetic optimality (this hope is however far from being implemented or even supported by concrete results to date). One can also remark that in the hands of physicists [see, in particular, D. F. Escande, Phys. Rep. 121 (1985), no. 3-4, 165–261; MR0791722] renormalization in the one-frequency case suggested astonishingly precise patterns and predictions, which have been numerically checked but remain far beyond what mathematicians are able to vindicate or sometimes even to state in a rigorous fashion.

As noted in the introduction to the present paper, a main difficulty in trying to implement renormalization in the multidimensional case has to do with the lack of a satisfactory multidimensional version of continued fractions. Renormalization is closely related to the use of simultaneous approximation, as opposed to small denominators, which pertain to linear approximation (approximation of linear forms). We note parenthetically that the use of simultaneous approximation has been introduced by the reviewer [Uspekhi Mat. Nauk 47 (1992), no. 6(288), 59–140; MR1209145; see also in Progress in nonlinear science, Vol. 1 (Nizhny Novgorod, 2001), 116–138, RAS, Inst. Appl. Phys., Nizhnii Novgorod, 2002; MR1965028] in Hamiltonian perturbation theory in a slightly different context and led to qualitative and drastic improvements of the results about long-time stability of near integrable Hamiltonian systems, as pioneered by N. N. Nekhoroshev. It also suggested new directions and phenomena. However, concerning the conservation of tori, one does need a detailed multidimensional continued fraction algorithm and this in itself constitutes a very hard problem. In fact, that problem has been shown to be in some sense not solvable. More precisely, G. Szekeres showed in the early seventies [Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 13 (1970), 113–140 (1971); MR0313198] that no multidimensional algorithm can possess all the wonderful properties of the ordinary one-dimensional continued fraction algorithm. They are simply incompatible in the multidimensional case. Yet, in the past three decades or so (partly perhaps under the impulse of V. I. Arnold), there has been a flurry of research on multidimensional continuous fractions, with a marked dynamical flavor. This led in particular to the development of an algorithm by J. C. Lagarias [Proc. London Math. Soc. (3) 69 (1994), no. 3, 464–488; MR1289860] which is used in the present paper and enjoys several nice properties.

Concretely speaking, the “renormalization algorithm” for proving Kolmogorov’s result
is in fact again quite close to the original scheme. One uses the same normal form except that at each step attention focuses not on the frequency of the sought-after invariant torus, but on a simultaneous (rational) approximation thereof. As a result, and this is the main point, no small divisors appear in the homological equation; but that difficulty is replaced by a necessary very careful control of the analyticity domains (which in the absence of small divisors govern the size of the various remainders). In other applications of renormalization, one can find a scaling pattern with relatively simple self-similarity properties, leading to “fractals” or “multifractals”. This is of course naive as soon as arithmetic comes into play and one has to be content with exploiting the translation of Siegel’s condition in order to prove a form of “iterative lemma” (here Theorem 3.6). As mentioned above the resulting proof is substantially more cumbersome than Kolmogorov’s original proof and does not so easily lend itself to the many known variants and improvements of the latter (which it would seem in any case rather pointless to repeat). It could aptly be termed a simultaneous approximation proof of Kolmogorov’s 1954 result on invariant tori.

In closing, let us return to the two related hopes that renormalization in general and the present paper in particular may foster. Both are briefly mentioned in its introduction. The first, already evoked, has to do with optimal or “natural” Diophantine conditions. At present the persistence of invariant tori had been proved under a weak Diophantine condition introduced by A. D. Bryuno (and a weak nondegeneracy condition); we refer in particular to the work of H. Rüssmann [Regul. Chaotic Dyn. 6 (2001), no. 2, 119–204; MR1843664]. Bryuno’s condition is however quite cumbersome and cast in terms of linear approximation (small divisors). It can be translated, at least in one direction, into a simultaneous condition [cf. F. Golse and P. Lochak, C. R. Acad. Sci. Paris Sér. I Math. 324 (1997), no. 9, 1047–1052; MR1451249], making it look more natural and close to the one-dimensional condition. Without even attacking optimality, for which radically new ideas seem to be required, it would perhaps be interesting to see whether renormalization methods can be developed to the point of proving the persistence of tori under such weak simultaneous Diophantine conditions.

The second direction is somehow related and in dynamical terms has to do with determining the most “robust” tori, i.e. those that break last when the perturbation parameter is increased. In dimension 1, the dynamical story is complicated and far from well-understood, but at least the arithmetic aspect is easy and well-known. The so-called “golden ratio” has plenty of elementary and well-known arithmetic properties, which make it an obvious candidate, for instance as the frequency of the “last surviving” invariant circle for the standard map. This is a global prediction in frequency space but of course one just as easily obtains local predictions which can and have been checked at least numerically. In many dimensions, the difficulties (really hard ones) start with the arithmetic and the mysterious story of the analogs of the golden ratio, that is the (locally) worst approximable vectors. Part of that story was recounted in [P. Lochak, op. cit.; MR1209145(§V.1)], leading at least to plausible predictions as to the locally most robust tori. It would certainly be interesting to refine and test such predictions using renormalization (and probably computers to start with). So all in all, having recovered in the present paper the integrality of Kolmogorov’s 1954 result using renormalization or simultaneous approximation, which here amounts to the same thing, hopefully will open the way to more refined and suggestive results.

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