This paper represents a significant advance in the long and important interaction between the ergodic theory of algebraic actions on homogeneous spaces and number theory. Raghunathan formulated a conjecture of the following shape. For a connected Lie group $G$, lattice $\Gamma \leq G$ and group $H \leq G$ generated by unipotents, the possible $H$-orbit closures and $H$-invariant ergodic probability measures on $G/\Gamma$ have algebraic descriptions (that is, exhibit topological or measurable “rigidity”). This should be viewed in contrast to the geodesic flow on $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$, which has a huge diversity of invariant measures.

A dramatic insight into the power of these ideas came with G. A. Margulis’ proof of the long-standing Oppenheim conjecture [in *Number theory, trace formulas and discrete groups (Oslo, 1987)*, 377–398, Academic Press, Boston, MA, 1989; MR0993328]; Margulis proved a special case of Raghunathan’s conjecture for the space $SL_3(\mathbb{R})/SL_3(\mathbb{Z})$ of unimodular lattices in $\mathbb{R}^3$ and applied this to solve the conjecture. The full conjecture was later shown by M. Ratner [Ann. of Math. (2) 134 (1991), no. 3, 545–607; MR1135878; Duke Math. J. 63 (1991), no. 1, 235–280; MR1106945]. An important feature of these actions is that individual elements of the action already exhibit rigidity.

A different source of rigidity comes from group actions in which each element behaves like a hyperbolic map—with a great diversity of invariant measures and nontrivial closed invariant subsets—but the action as a whole exhibits rigidity. This was first observed in work of H. Furstenberg [Math. Systems Theory 1 (1967), 1–49; MR0213508], who showed that the natural action $t \mapsto ax \pmod{1}$ of the semi-group generated by two multiplicatively independent natural numbers on the circle $\mathbb{R}/\mathbb{Z}$ has no nontrivial closed invariant sets. He raised the natural question of whether this system would also exhibit measure rigidity—specifically, could it have any non-atomic ergodic invariant measures other than Lebesgue measure? Partial results on this question and related generalizations by D. J. Rudolph [Ergodic Theory Dynam. Systems 10 (1990), no. 2, 395–406; MR1062766], A. S. A. Johnson [Israel J. Math. 77 (1992), no. 1-2, 211–240; MR1194793], B. Host [Israel J. Math. 91 (1995), no. 1-3, 419–428; MR1348326], and others led to the formulation of far-reaching conjectures by Margulis [in *Mathematics: frontiers and perspectives*, 161–174, Amer. Math. Soc., Providence, RI, 2000; MR1754775], by Furstenberg, and by Katok and R. J. Spatzier [Ergodic Theory Dynam. Systems 16 (1996), no. 4, 751–778; MR1406432]. A special case of Margulis’ conjecture concerns actions of the group $A$ of positive diagonal matrices in $SL_k(\mathbb{R})$ for $k \geq 3$ on the space $SL_k(\mathbb{R})/SL_k(\mathbb{Z})$: if $\mu$ is an $A$-invariant ergodic probability measure on this space, is there a closed connected group $L > A$ for which $\mu$ is the unique $L$-invariant measure on a single closed $L$-orbit (that is, $\mu$ is algebraic)?

An important example, due to M. Rees (unpublished), illustrates how delicate some of these phenomena are: there are lattices $\Gamma < SL_k(\mathbb{R})$ with $A$-invariant ergodic probability measures on $SL_k(\mathbb{R})/\Gamma$ that are not algebraic. These arise from the presence of $A$-invariant homogeneous subspaces with factors on which the action degenerates to a
In this paper the conjecture of Margulis is proved under the additional hypothesis that the measure $\mu$ gives positive entropy to some one-parameter subgroup of $A$. Results of Lindenstrauss and B. Weiss [Ergodic Theory Dynam. Systems 21 (2001), no. 5, 1481–1500; MR1855843] giving a complete classification of the algebraic measures are then used to deduce that under the same hypotheses, the measure $\mu$ is not compactly supported, and if $k$ is prime then $\mu$ is the unique measure on $\text{SL}_k(\mathbb{R})/\text{SL}_k(\mathbb{Z})$ invariant under the action of $\text{SL}_k(\mathbb{R})$.

This advance in our understanding of measure rigidity finds an important application in number theory. Writing $\langle t \rangle$ for the distance from $t \in \mathbb{R}$ to the nearest integer, Littlewood conjectured that

$$\liminf_{n \to \infty} n \langle nx \rangle \langle ny \rangle = 0$$

for every $x, y \in \mathbb{R}$. Notice that the continued fraction expansion of either one of $x$ or $y$ shows that the limit infimum is no more than $\frac{1}{2}$; Littlewood’s conjecture asks if a better sequence of integers can be found using knowledge of both $x$ and $y$. J. W. S. Cassels and H. P. F. Swinnerton-Dyer [Philos. Trans. Roy. Soc. London. Ser. A. 248 (1955), 73–96; MR0070653] proved Littlewood’s conjecture under the assumption that $x$ and $y$ lie in the same cubic number field.

It is not difficult to show that the set $\Xi$ of $(x, y)$ for which (1) fails is a Lebesgue null set; on the other hand if $x$ is badly approximable then $\liminf_{n \to \infty} n \langle nx \rangle > 0$ and the set of such numbers has Hausdorff dimension one. A. D. Pollington and S. L. Velani [Acta Math. 185 (2000), no. 2, 287–306; MR1819996] showed that for all $x$, the set of badly approximable $y$ for which (1) holds has Hausdorff dimension one.

In this paper the result on measure rigidity is applied to prove that $\Xi$ is a countable union of compact sets of box dimension zero. This is dramatic progress on Littlewood’s conjecture; removing the entropy hypothesis from the measure rigidity result obtained would prove the full conjecture. Results are also found on the generalization of Littlewood’s conjecture to higher dimensions.

The proof of the main measure rigidity result builds on earlier work substantially. A striking aspect of the ideas developed here is that two rather different approaches play a complementary role. One of these, developed by Einsiedler and Katok [Comm. Pure Appl. Math. 56 (2003), no. 8, 1184–1221; MR1989231; Israel J. Math. 148 (2005), 169–238; MR2191228], applies with $\text{SL}_k(\mathbb{Z})$ replaced by any discrete subgroup, and applies to the situation in which many one-parameter subgroups give positive entropy to the unknown $A$-invariant measure; these methods may be applied to show that $\Xi$ has Hausdorff dimension at most 1. The second, developed by Lindenstrauss [Ann. of Math. (2) 163 (2006), no. 1, 165–219; MR2195133] in his work on the quantum unique ergodicity conjecture, applies to the situation where few one-parameter subgroups have positive entropy.

It is impossible in a short review to convey the richness of ideas exploited and developed in this exceptionally interesting paper.

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Note: This list, extracted from the PDF form of the original paper, may contain data conversion errors, almost all limited to the mathematical expressions.

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