A nonlinear generalization of Perron’s theorem. (Russian)


By the famous Perron’s theorem, for any strictly positive matrix $A: \mathbb{R}_+^d \to \mathbb{R}_+^d$ we have $\|A^n x\|^{-1} A^n x - x^* \| \leq K \beta^n$, where $K > 0$, and $0 < \beta < 1$ are constants. The paper under review concerns its nonlinear generalization. Let $\alpha: \mathbb{R}_+^d \to \mathbb{R}_+^d$ be a functional satisfying: A1) $\alpha(\lambda x) = \lambda \alpha(x)$ for all $\lambda \geq 0$ and $x \in \mathbb{R}_+^d$, A2) $(\alpha'(x))^T x > 0$ for all nonzero $x \in \mathbb{R}_+^d$, and A3) $\alpha \in C^3$. Given a (nonlinear) mapping $F: \mathbb{R}_+^d \to \mathbb{R}_+^d$ define $T_x = [\alpha(F(x))]^{-1} F(x)$. $T$ is considered as a transformation of $S$, where $S = \{x \in \mathbb{R}_+^d : \alpha(x) = 1\}$ is endowed with the projective metric $\rho$. In the main result of the paper (Theorem 3) it is proved that whenever the following condition $(\ast)$ $A(x) = F'(x)(I - x \gamma(x)^T) + F(x) \gamma(x)^T \geq B > 0$ is fulfilled (where $\gamma(x) = [\alpha'(x)^T x]^{-1} \alpha'(x)$ and $I$ is the identity matrix) then there exist constants $N = N(F, \alpha)$, $0 < \lambda = \lambda(F, \alpha) < 1$ such that $\rho(T^n x, T^n y) < \lambda^r(x, y)$ for all $n > N$ and $x, y \in S$. It follows that there is a unique fixed point $x^* \in S$ such that $\sup_{x \in S} \|T^n x - x^*\| \leq K \beta^n$ for some constants $K > 0$, $0 < \beta < 1$. An important class of mappings $F$ satisfying $(\ast)$ is described by Theorem 4. It appears that $(\ast)$ holds if $\alpha'(x) \geq 0$, $F(A x + (1 - \lambda) y) \geq \lambda F(x) + (1 - \lambda) F(y)$, $F'(x) > 0$ for all $0 \leq \lambda \leq 1$ and $x, y \in \mathbb{R}_+^d$, and $F(x) \leq F(y)$ whenever $x \leq y \in \mathbb{R}_+^d$. Besides the main topic several interesting remarks on extensions of Perron’s theorem in other directions are included. Nonlinear extensions of Perron’s theorem were obtained in the past by U. Krause [compare Nonlinear Anal. 47 (2001), no. 3, 1457–1466; MR1977031; J. Math. Econom. 15 (1986), no. 3, 275–282; MR0871157].

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