Numerous problems in ergodic theory and dynamical systems are related to solving cohomological equations, i.e., given a flow $\phi^t$ and a function $g$, find a function $f$ such that $g = (d/dt) f \circ \phi^t |_{t=0}$. Any recurrence in the flow clearly obstructs the existence of solutions for some $g$, and those $g$ for which there are solutions are said to be coboundaries. In two complementary situations the obstructions have been well-described by the set of invariant measures. For a linear flow on the torus with Diophantine rotation number any sufficiently regular $g$ with mean zero is a coboundary, i.e., the unique invariant measure provides the sole obstruction. For Anosov flows the Livšic Theorem tells us that a Hölder continuous $g$ whose integral over any periodic orbit vanishes is a coboundary. Since periodic points generate the set of invariant measures, these again provide the full set of obstructions. A related issue is the interplay between the regularity of $g$ and that of $f$; in the former case this is determined by the Diophantine parameter.

Forni found that area-preserving flows on surfaces of higher genus have additional obstructions to solving cohomological equations that are distributions (in the sense of generalized functions), and these additional obstructions give insights into, for example, ergodic averages of functions [Ann. of Math. (2) 146 (1997), no. 2, 295–344; MR1477760; Ann. of Math. (2) 155 (2002), no. 1, 1–103; MR1888794]. The present paper studies these same issues for horocycle flows (on the unit tangent bundle of a Riemann surface $M$ of finite area), which are also “parabolic” in a way similar to flows on surfaces of higher genus. Here, too, there are additional obstructions, and the authors study their Sobolev regularity and construct smooth solutions. This yields analogous information about ergodic averages. For example, unlike in the case of hyperbolic flows, the central limit theorem fails here. To give a flavor, a complete set of obstructions to existence of smooth solutions of the cohomological equation for $g \in W^s(SM)$ (the $s$-Sobolev space of functions on the unit tangent bundle) is given by the space of distributions in $W^{-s}(SM)$ that are invariant under the horocycle flow. This space is completely described; the first theorem describes the (countably infinite-dimensional) space of invariant distributions dual to $C^\infty(SM)$ completely in terms of the spectrum of the Laplace-Beltrami operator and the genus of $M$.

There is also a description of properties of the action of the geodesic flow on this space of distributions.

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References

4. W. H. Barker, $L^p$ harmonic analysis on $\mathfrak{sl}(2, \mathbb{R})$, Mem. Amer. Math. Soc. 76 (1988),


44. O. S. Parasyuk, *Flows of horocycles on surfaces of constant negative curvature* (in Russian), Uspekhi Mat. Nauk 8, no. 3 (1953), 125–126. MR0058883


Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

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