If \( \mathcal{H} \) is a complex infinite-dimensional separable Hilbert space, then let \( \mathcal{L}(H) \) stand for the algebra of bounded linear operators on \( \mathcal{H} \). An operator \( T \) in \( \mathcal{L}(H) \) is said to be subnormal if there exist a Hilbert space \( \mathcal{K} \) containing \( \mathcal{H} \) and a normal operator \( N \) in \( \mathcal{L}(K) \) such that \( \mathcal{H} \) is invariant for \( N \) and \( T \) is the restriction of \( N \) to \( \mathcal{H} \). Letting \([A, B]\) stand for the commutator \( AB - BA \) of two operators \( A, B \) in \( \mathcal{L}(H) \), one says that \( T \) in \( \mathcal{L}(H) \) is \( k \)-hyponormal if the \( k \times k \) operator matrix having the \((i, j)\)th entry \([T^i, T^j]\) is positive. Given a bounded sequence \( \{\alpha_n\}_{n \geq 0} \) of positive reals, one can define a weighted shift operator \( W_\alpha \) (with the weight sequence \( \{\alpha_n\}_{n \geq 0} \) via the relations \( W_\alpha e_n = \alpha_n e_{n+1} \), where \( \{e_n\}_{n \geq 0} \) is an orthonormal basis for \( \mathcal{H} \). Given a finite ordered \((k + 1)\)-tuple \( \alpha = (\alpha_0, \ldots, \alpha_k) \) of positive reals, one says that a sequence \( \{\hat{\alpha}_n\}_{n \geq 0} \) of positive reals is recursively generated by \( \alpha \) if \( \hat{\alpha}_j = \alpha_j \) for \( 0 \leq j \leq k \) and there exist \( r \geq 1 \) and reals \( \phi_0, \ldots, \phi_{r-1} \) such that \( \gamma_{n+r} = \phi_0 \gamma_0 + \cdots + \phi_{r-1} \gamma_{n+r-1} \) for all \( n \geq 0 \), where \( \gamma_0 = 1 \) and \( \gamma_n = \gamma_{n-1} \hat{\alpha}_{n-1}^2 \) \((n > 1)\); in such a case \( W_\hat{\alpha} \) itself is said to be recursively generated (by \( \alpha \)). An \( n \)-step extension of a weighted shift operator \( W_\hat{\alpha} \) recursively generated by \( \alpha \) is a weighted shift operator that has as its weight sequence the augmented sequence obtained from \( \{\hat{\alpha}_n\}_{n \geq 0} \) by inserting \( n \) positive reals before \( \hat{\alpha}_0 \). The so-called “rank-1 perturbation” of a recursively generated weighted shift operator is obtained by changing one term of the associated weight sequence; it is not difficult to see that, for any rank-1 perturbation of a recursively generated weighted shift operator, the associated weight sequence is an \( n \)-step extension of a recursively generated sequence for some \( n \geq 0 \).

The authors establish a few results relating the requirement of subnormality for an \( n \)-step extension of a recursively generated weighted shift operator (and, in particular, of rank-1 perturbation of a recursively generated weighted shift operator) to the requirement of \( k \)-hyponormality for such an operator. In particular, they show that a 1-step extension of a weighted shift operator recursively generated by a \((k + 1)\)-tuple \( \alpha \) of positive reals is subnormal if and only if it is \(((k + 1)/2 + 1)\)-hyponormal (with \([m] \) denoting the integral part of a positive integer \( m \)), and that an \( n \)-step extension, with \( n > 1 \), is subnormal if and only if it is \(((k + 1)/2 + 2)\)-hyponormal.

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**References**


Note: This list, extracted from the PDF form of the original paper, may contain data conversion errors, almost all limited to the mathematical expressions.

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