Invariant measures for Burgers equation with stochastic forcing.


FEATURED REVIEW.

The existence, uniqueness and ergodic properties of invariant measures for evolution equations perturbed by random noise have been studied in numerous papers that appeared in recent years. In the present paper the authors study the Burgers equation of the form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left( u^2 \right) = \epsilon \frac{\partial^2 u}{\partial x^2} + f(x, t)$$

where $f(x, t) = \frac{\partial F}{\partial x}(x, t)$ denotes a random forcing function periodic in $x$ with period 1, which is a white noise in $t$. The potentials take the form

$$F(x, t) = \sum_{k=1}^{\infty} F_k(x) \dot{B}_k(t),$$

where $(B_k)$ is a sequence of independent standard scalar Wiener processes defined on $\mathbb{R}^1 \times (\Omega, \mathcal{F}, P)$ and $(\Omega, \mathcal{F}, P)$ is a given probability space. The functions $F_k$ are supposed to be periodic with period 1 and such that $f_k = F_k' \in C^r(S^1), \|f_k\|_{C^r} \leq Ck^{-2}$ for some $r \geq 3$, where $S^1$ denotes the unit circle and $C$ is a generic constant. Most of the results in the paper, however, concern the “nonviscous” case $\epsilon = 0$, i.e. the equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left( u^2 \right) = f(x, t).$$

The main goal of the paper is to establish existence and uniqueness of an invariant measure for the equation (3). A detailed description of the structure and regularity properties for solutions to (3) living on the support of this measure is also given and convergence of the invariant functionals (and invariant measures) defined by the equation (1) as $\epsilon \to 0$ is studied.

The Burgers equations perturbed by random noise serve as a suitable model for a wide range of problems in statistical physics. Most recently, they have provided the field-theoretic techniques in hydrodynamics [see, e.g., T. Gotoh and R. H. Kraichnan, Phys. Fluids 10 (1998), no. 11, 2859–2866; MR1650782; A. M. Polyakov, Phys. Rev. E (3) 52 (1995), no. 6, part A, 6183–6188; MR1384741]. The authors also give references illustrating applicability of the model elsewhere (study of vortex lines in superconductors, charge density waves, directed polymers, and some others).

Let us describe the contents of the paper in more detail. At first glance it looks rather surprising that although the viscosity term is absent in the equation (3) ($\epsilon = 0$), and energy is constantly supplied to the system, the invariant measure does exist. The energy dissipation that is needed to this end is provided by an additional entropy condition, which reads $u(x+, t) \leq u(x-, t)$ for all $(x, t)$. In Lemma 2.1 it is proved that for each initial datum $u_0$ from the Skorokhod space $D$ on $S^1$ there exists a unique solution to (3) living in $D$ for $t \geq t_0$ P-a.s., satisfying the entropy condition and the
initial condition \( u(x, t_0) = u_0(x) \). Furthermore, the solution is given by

\[
(4) \quad u(x,t) = \frac{\partial}{\partial x} \inf_{\xi(t)=x} \left\{ A_{t_0,t}(\xi) + \int_0^{\xi(t_0)} u_0(z) \, dz \right\}
\]

and \( u(\cdot, t) \in D \), where the infimum is taken over all Lipschitz continuous curves \( \xi \) on \([t_0, t] \) satisfying \( \xi(t) = x \) and

\[
(5) \quad A_{t_1,t_2}(\xi) = \int_{t_1}^{t_2} \left\{ \frac{1}{2} \dot{\xi}(s)^2 - \sum_k f_k(\xi(s)) \dot{\xi}(s)(B_k(s) - B_k(t_1)) \right\} \, ds
\]

\[
+ \sum_k F_k(\xi(t_2))(B_k(t_2) - B_k(t_1)).
\]

This variational characterization of solutions to (3) satisfying the entropy condition (which is well known in classical cases) is the heart of the method used in the paper. The key object needed in the construction of an invariant measure is the so-called one-sided minimizer \( \gamma_{x,t}(\cdot) \), which is, roughly speaking, a curve minimizing the action (5) over the interval \((-\infty, t]\), coming to \((x, t) \). In Section 3 existence and uniqueness of one-sided minimizers are proved. While existence is obtained by a limiting procedure taking limits of minimizers over finite intervals \([-k, 0]\), uniqueness follows from their intersection properties. It is interesting to note that while absence of two intersections is a general fact, the authors prove that minimizers have no intersections at all (except for the situation when they intersect at the terminal point). Setting \( u^\omega(x, t) = \inf \gamma_{x,t}(t) \), where the infimum is taken over all such minimizers coming to \((x, t) \), it is shown that the random field \( u^\omega(\cdot, 0) \) defines a stationary Markov solution to the equation (3). Uniqueness of the invariant measure is closely related to uniqueness of one-sided minimizers. This method may be viewed as a “pathwise” approach because the authors’ basic strategy is to show the following “one force, one solution” principle: For almost all \( \omega \) there exists a unique solution to (3) defined on the whole real line. (On a more general level, a similar approach is used to study random dynamical systems and random attractors [see, e.g., H. Crauel and F. Flandoli, Probab. Theory Related Fields 100 (1994), no. 3, 365–393; MR1305587] in Sections 5–7 properties of solutions supported by the invariant measures are studied in detail. The main object is the two-sided minimizer (and a dual object called the main shock) which is defined similarly to the one-sided minimizer but for the whole \( \mathbb{R}^1 \). The two-sided minimizers exist and are unique under a nondegeneracy condition on the forcing \( F_k \): If \( x^* \) is a local maximum of some \( F_k \) denote by \( I(x^*) \) a closed interval on \( \mathbb{S}^1 \) containing \( x^* \) such that, on \( I(x^*) \), \( f_k < 0 \) and \( f_k > 0 \) to the right of \( x^* \) and to the left of \( x^* \), respectively. It is supposed that there exists a finite set of points \( x^* \) each of which is a local maximum of some \( F_k \) with the following property: for any \((x_1, x_2) \in \mathbb{S}^1 \times \mathbb{S}^1 \) there exists an \( x^* \) from this finite set such that \( x_1, x_2 \in I(x^*) \). Then the two-sided minimizer is shown to be a hyperbolic trajectory of the dynamical system corresponding to the characteristics of (3), that is,

\[
(6) \quad \frac{dx}{dt} = u, \quad \frac{du}{dt} = \frac{\partial F}{\partial x} (x, t).
\]

This allows the authors to study stable and unstable manifolds of the two-sided minimizer and regularity of solutions to the original equation. Finally, in Section 8 the authors show weak convergence of the invariant measures of the equation (1) (whose existence has been proved earlier by Sinai) to the invariant measure of (3) as \( \epsilon \to 0 \). The main tool used in the proof is the Hopf-Cole transformation.

In the reviewer’s opinion, a significant contribution in the field has been provided in the paper. The proofs of the main results are often based on “hard” analysis using the
special structure of the equation, which makes them highly nontrivial.

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References


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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.