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Differential rigidity of Anosov actions of higher rank abelian groups and algebraic lattice actions. (English. English summary)


From the introduction: “In this paper we bring the study of the local differentiable rigidity of algebraic Anosov actions of $\mathbb{Z}^k$ and $\mathbb{R}^k$ on compact manifolds as well as orbit foliations of such actions, started in a number of previous works, to a near conclusion. Our results for the abelian group actions formulated in Section 2.1 allow us to obtain comprehensive local $C^\infty$-rigidity for two very different types of algebraic actions of irreducible lattices in higher-rank semisimple Lie groups: (i) the Anosov actions by automorphisms of tori and nilmanifolds (Theorem 15, Section 3), and (ii) the actions on Furstenberg boundaries, in particular projective spaces (Theorem 17, Section 4). While the latter area was virtually unapproachable save for a special result by M. Kanai (Geom. Funct. Anal. 6 (1996), no. 6, 943–1056; MR1421873), the former was extensively studied, beginning with [S. E. Hurder, Ann. of Math. (2) 135 (1992), no. 2, 361–410; MR1154597; A. B. Katok and J. W. Lewis, Israel J. Math. 75 (1991), no. 2-3, 203–241; MR1164591] and continuing in subsequent papers. Our result (Theorem 15 in Section 3) substantially extends all these earlier works and brings the question to a final solution. It also gives further credence to the global conjecture that all Anosov actions of such lattices are smoothly equivalent to one of the above. Such a classification was established for the much more special class of Cartan actions in [E. Goetze and R. J. Spatzier, “Smooth classification of Cartan actions of higher rank semisimple Lie groups and their lattices”, Preprint, Univ. Michigan, 1996; per bibl.].

“The objects considered in this paper, i.e., group actions and foliations, are assumed to be of class $C^\infty$. Accordingly, the basic notions, including structural stability, are adapted to this case, while they sometimes appear more naturally for lower regularity. In fact, for each of our results a certain finite degree of regularity is sufficient, resulting in a loss of regularity in the conjugating diffeomorphisms. We leave the detailed study of optimal regularity conditions to a later paper. In proper places we make specific comments about the degree of regularity sufficient to guarantee existence of $C^1$-conjugating diffeomorphisms.

“Let $M$ be a compact manifold. We call two foliations $\mathcal{F}$ and $\mathcal{G}$ on $M$ orbit equivalent if there is a homeomorphism $\phi: M \to M$ such that $\phi$ takes the leaves of $\mathcal{F}$ to those of $\mathcal{G}$. We call $\phi$ an orbit equivalence. If $\phi$ is a $C^\infty$-diffeomorphism, we call $\mathcal{F}$ and $\mathcal{G}$ $C^\infty$-orbit equivalent.

“Endow the space of foliations on $M$ with the $C^1$-topology, i.e., two foliations are close if their tangent distributions are $C^1$-close. We call a $C^\infty$-foliation $\mathcal{F}$ structurally stable if there is a neighborhood $U$ of $\mathcal{F}$ such that any foliation in $U$ is orbit equivalent to $\mathcal{F}$. We call a $C^\infty$-foliation $\mathcal{F}$ $C^\infty$-locally rigid if there is a neighborhood $U$ of $\mathcal{F}$ such that any $C^\infty$-foliation in $U$ is $C^\infty$-orbit equivalent to $\mathcal{F}$.

“Call a $C^\infty$-action of a finitely generated discrete group $\Gamma$ structurally stable if any $C^\infty$-perturbation of the action which is $C^1$-close on a finite generating set is conjugate to it via a homeomorphism. Call such a $\Gamma$-action locally $C^\infty$-rigid if any perturbation of the action which is $C^1$-close on a finite generating set is conjugate to the original
action via a $C^\infty$-diffeomorphism. We say that two actions of a group $G$ agree up to an automorphism if the second action can be obtained from the first one by composition with an automorphism of the underlying group. Call a $C^\infty$-action of a Lie group $G$ locally $C^\infty$-rigid if any perturbation of the action which is $C^1$-close on a compact generating set is $C^\infty$-conjugate up to an automorphism. The notions of $C^r$, $r \geq 1$, local rigidity for foliations and actions are defined accordingly.

For $\Gamma = \mathbb{Z}$ or $\mathbb{R}$, i.e., for the classical dynamical systems, diffeomorphisms ($\mathbb{Z}$-actions) and flows ($\mathbb{R}$-actions), local $C^\infty$-rigidity never takes place. Moreover, the orbit foliations of flows are not locally $C^\infty$-rigid either. In those cases, it does not help to allow the conjugacy in the definitions to be only a $C^1$-diffeomorphism. Structural stability, on the other hand, is a rather widespread (although not completely understood) phenomenon. Since local $C^\infty$-rigidity implies structural stability, one can immediately see from the necessary conditions for structural stability that there are always moduli of $C^1$-conjugacy in the structurally stable case; this shows the impossibility of differentiable rigidity.

Certain structural stability results, namely those by Hirsch, Pugh and Shub, are important for our purposes. They established structural stability of central foliations of certain partially hyperbolic dynamical systems, which implies, in particular, that the orbit foliations of hyperbolic (Anosov) actions of $\mathbb{R}^k$ are structurally stable. At the beginning of Section 2.2 we explain how this fact is used as the starting point in our proof of local differentiable rigidity.

Contrary to the classical case, in the higher-rank situation of a $\mathbb{Z}^k$- or $\mathbb{R}^k$-action for $k \geq 2$, $C^\infty$-local rigidity is possible and in fact seems to be as closely related to the hyperbolic behavior as structural stability is in the classical case. Existence of this phenomenon was first discovered in [A. Katok and J. Lewis, op. cit. (Theorem 4.2)] for certain (in a sense ‘maximal’) actions of $\mathbb{Z}^k$ by toral automorphisms and was extended in [A. Katok and R. J. Spatzier, “Differential rigidity of hyperbolic abelian actions”, Preprint, Math. Sci. Res. Inst., Berkeley, CA, 1992; per bibl.] to a broader class of Anosov actions by both $\mathbb{R}^k$ and $\mathbb{Z}^k$ (multiplicity-free standard actions). The results on cocycle rigidity which first appeared in [A. Katok and R. J. Spatzier, op. cit., 1992] are fairly definitive; they appeared in print as [A. B. Katok and R. J. Spatzier, Inst. Hautes Études Sci. Publ. Math. No. 79 (1994), 131–156; MR1307298; Math. Res. Lett. 1 (1994), no. 2, 193–202; MR1266758]. On the other hand, the local rigidity result for actions in [A. Katok and R. J. Spatzier, op. cit., 1992] was still too restrictive; in particular, it covered the key situation of the Weyl chamber flows only for split semisimple groups. In the present paper we obtain much more general rigidity results which are close to being definitive. We prove $C^\infty$-local rigidity of most known irreducible Anosov actions of $\mathbb{Z}^k$ and $\mathbb{R}^k$ as well as the orbit foliations of the latter (Theorem 1, Corollaries 2 through 5). All such actions are essentially algebraic (see Section 2.1), and the remaining technical assumption of semisimplicity of the linear part of the action is satisfied in all interesting cases, including the Weyl chamber flows, which appear in applications to the rigidity of actions of irreducible cocompact lattices in higher-rank semisimple Lie groups on Furstenberg boundaries (Section 4), and those actions by automorphisms of tori and nilmanifolds which are needed for the proof of rigidity of actions of irreducible lattices in higher-rank semisimple Lie groups by such automorphisms (Section 3).”

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