von Neumann, John

★Mathematical foundations of quantum mechanics. (English summary)
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FEATURED REVIEW.
Many mathematics researchers believe that John von Neumann and David Hilbert were the two greatest mathematicians of the twentieth century. Even those who have other favorite candidates will probably agree that von Neumann is among the top five of such a list. Unlike most mathematics investigators, we can easily gain a glimpse of von Neumann’s personality from his writings. Here was a playful man, a man who was bold and fearless. He was not afraid to experiment, to calculate and to reveal his thought processes in public. In the late 1950s and early 1960s when the reviewer and some of his colleagues were in graduate school, von Neumann’s book was the only one available (in English) that presented quantum mechanics in a rigorous mathematical fashion. And what a delight it was. It was insightful, motivated, intuitive and understandable physics and it was mathematically consistent. When we took courses in functional analysis, we found that we were seeing the same things over again. Quantum mechanics was Hilbert space analysis and, conversely, much of Hilbert space analysis was quantum mechanics. It was exciting to see that such an elegant, abstract field of mathematics was applicable to revolutionary studies of the natural world. Quantum mechanics was the greatest scientific revolution since Newtonian mechanics. Hilbert space theory was to quantum mechanics as the calculus was to Newtonian mechanics, and von Neumann held the leading beacon of light. Other rigorous books on quantum mechanics and quantum field theory appeared in the mid 1960s such as G. W. Mackey’s The mathematical foundations of quantum mechanics: a lecture-note volume [Benjamin, New York, 1963; MR0155567] and R. F. Streeter and A. S. Wightman’s PCT, spin and statistics and all that [Benjamin, New York, 1964; MR0161603]. But as good as these books were, they seemed to lack some of the spark of von Neumann’s text. We now briefly summarize the contents of this book.

In the preface, von Neumann begins: “The object of this book is to present the new quantum mechanics in a unified representation which, so far as it is possible and useful, is mathematically rigorous.” He goes on to explain that his treatment gives a mathematically consistent alternative to that of P. A. M. Dirac’s The principles of quantum mechanics [Milford, London, 1930; JFM 56.0745.05]. His exposition is a model of succinctness and clarity. For example, the following is taken from his writings in Chapter I. “Quantum theory has made it clear beyond doubt that all elementary processes, that is, all occurrences of an atomic or molecular order of magnitude, obey the ‘discontinuous’ laws of quanta…. Of great significance is the fact that the general opinion in theoretical physics has accepted the idea that the principle of continuity prevailing in the perceived macroscopic world, is merely simulated by an averaging process in a world which in turn is discontinuous by its very nature. This simulation is such that man generally perceives the sum of many billions of elementary processes...
simultaneously, so that the leveling law of large numbers completely obscures the real nature of the individual processes.”

Chapter I. Introductory considerations. Von Neumann formulates the two theories of quantum mechanics, the Heisenberg matrix mechanics and Schrödinger’s wave mechanics, and shows their equivalence. First he considers the Dirac-Jordan method of showing equivalence. This he dismisses as being nonrigorous because of its reliance on delta functions and their derivatives, which he calls “mathematical fictions”. He then passes to Schrödinger’s method using Hilbert space techniques. He employs the Riesz-Fischer theorem and motivates the need for studying abstract Hilbert spaces.

Chapter II. Abstract Hilbert spaces. Von Neumann defines a (separable) Hilbert space and gives a (now) standard treatment of Hilbert space geometry. He then discusses (linear) operators in Hilbert space, with a special emphasis on projections. He stresses that operators need not be continuous although their domains are always assumed to be dense linear manifolds. Next he states the eigenvalue problem, which has central importance in quantum mechanics. Given a self-adjoint (hypermaximal) operator $H$, show that $H$ possesses a complete orthonormal set of eigenvectors. Von Neumann points out that although this is possible for finite-dimensional Hilbert spaces, it is not possible in general. He then replaces this problem with one that is equivalent in finite dimensions but is solvable in infinite dimensions. In the finite-dimensional case, the eigenvalue problem is equivalent to diagonalizing $H$. He shows that this latter problem is equivalent to finding a spectral family $E(\lambda)$ such that $H$ can be written as a Stieltjes integral relative to $E(\lambda)$. This problem can be solved in infinite dimensions and the solution was worked out previously by the author [Math. Ann. 102 (1929), 49–131; JFM 55.0824.02; Math. Ann. 102 (1929), 370–427; JFM 55.0825.02; Math. Ann. (2) 32 (1931), 191–226; Zbl 002.26703]. The methods and terminology are, for the most part, still used today. Great attention is placed on motivation, and detailed calculations and examples are given to show what holds and what does not hold. Von Neumann then establishes a functional calculus for self-adjoint operators and discusses commuting self-adjoint operators. He next presents his famous theorem that if $A$ and $B$ are commuting self-adjoint operators, then they are both functions of a single self-adjoint operator. This is one of the many cases in which he reveals his thought processes and his willingness to explore various possibilities. In particular, he frequently first gives a heuristic argument without mathematical rigor and then shows how the argument can be made rigorous. This is how von Neumann worked and how he has written this book. One is given a firsthand view of how the master thinks. The road may not be the shortest but we always know where he is headed. In the last section of this chapter, he discusses the trace of operators.

Chapter III. The quantum statistics. Von Neumann begins by postulating that quantum observables correspond to self-adjoint operators on a Hilbert space and (pure) states correspond to unit vectors. Applying the spectral theory developed in Chapter II, von Neumann announces his famous statistical postulate for quantum mechanics. If $A_1, \cdots, A_n$ are commuting self-adjoint operators corresponding to observables and $E_1, \cdots, E_n$ are their respective spectral measures, then for any state corresponding to a unit vector $\phi$, the probability that these observables have simultaneous values in Borel sets $B_1, \cdots, B_n$ is (1) \[ \|E_1(B_1) \cdots E_n(B_n)\phi\|^2. \] He shows that (1) is equivalent to the fact that if an observable corresponds to the self-adjoint operator $A$ and $\phi$ is in the domain of $A$, then the expectation of the observable in the state $\phi$ is $\langle A\phi, \phi \rangle$. He then discusses the statistical ramifications of this postulate. Unlike classical mechanics, in which the state of a physical system determines the values of all physical quantities associated with the system, in quantum mechanics the state only determines their probability distributions. Nevertheless, from a given initial state, subsequent states can be calculated causally
from Schrödinger’s time-dependent wave equation. Why do the operators $A_1, \ldots, A_n$ have to commute? One reason is that the probability (1) would not be additive, otherwise. Von Neumann observes that this deficiency may be caused by an incompleteness in (1) and that there may exist a more general formula, containing (1) as a special case. He then shows that such a generalization of (1) is not possible. In particular, he shows that quantum observables are simultaneously measurable with arbitrarily high accuracy if and only if their corresponding self-adjoint operators commute. He next discusses the uncertainty relations that result when two observables with corresponding noncommuting operators are measured. The interpretation of projections as quantum propositions is developed next. (A later paper with G. Birkhoff [Ann. of Math. (2) 37 (1936), 823–843; Zbl 015.14603] expanded on this topic to create the theory of “quantum logic”.)

The last section of this chapter is devoted to radiation theory. In this section he shows how transition probabilities between stationary states of an atomic electron can be derived from the statistical postulate. In order to discuss this completely, the theory of electromagnetic radiation is first treated classically and then quantum mechanically. In this treatment, von Neumann elaborates on the earlier work of Dirac. From this development, we witness the beginning phases of quantum electrodynamics. The reasoning is long and involved (forty pages of dense calculations) and is today replaced by the more elegant methods of quantum field theory. But it is quite instructive to see these early calculations and it shows how one of the most difficult paradoxes of quantum theory, namely the dual nature of light as waves and particles, is brilliantly resolved.

Chapter IV. Deductive development of the theory. As shown in Chapter III, the statistical postulate is equivalent to (2) $\text{Exp}(A, \phi) = \langle A\phi, \phi \rangle$, where $\text{Exp}(A, \phi)$ is the expectation of $A$ in the state $\phi$. Von Neumann now shows that this formula can itself be derived from a few general qualitative assumptions. He argues that the expectation function $\text{Exp}(A)$ must have linearity and positivity properties and, assuming these properties, he shows that there exists a statistical operator (positive, trace-class operator) $T$ such that (3) $\text{Exp}(A) = \text{tr}(TA)$. This gives a generalization of (2) for mixed states, whereas (2) is formulated for pure (extremal) states. (Various investigators have criticized von Neumann’s assumption that $\text{Exp}(A)$ should be linear for all observables because this includes observables that may not be simultaneously measurable. However, in 1957, A. M. Gleason [J. Math. Mech. 6 (1957), 885–893; MR0096113] proved that (3) holds under the assumption of linearity only for simultaneously measurable observables.) Von Neumann then shows that there are no dispersion-free (zero variance) expectation functions and that the only extremal expectation functions are those of the form (2). Next he gives an argument against a hidden-variable theory for quantum mechanics. In such a theory, it is assumed that a state $\phi$ does not give a complete description of reality. To obtain a complete description, one must augment $\phi$ with additional parameters or variables to obtain a complete state. Once this complete state is known dispersions vanish and the theory becomes deterministic. The quantum probabilities given by $\phi$ are obtained by averaging over the hidden variables. Von Neumann argues that if a hidden-variable description existed for quantum mechanics, then the present quantum theory must be objectively false and this is contrary to experiment. This constituted the first no-go theorem for hidden variables. (Some of the assumptions of this theorem have been criticized for physical reasons by various researchers and hidden-variable theories have been proposed. Other no-go theorems have been proved under weaker assumptions, including J. S. Bell’s famous inequalities [Rev. Modern Phys. 38 (1966), 447–452; MR0208927]. However, the argument over hidden variables in quantum mechanics still rages today [E. G. Beltrametti and G. Cassinelli, The logic of quantum mechanics, Addison-Wesley Publishing Co., Reading, Mass., 1981; MR0635780].) The chapter concludes with a discussion of mixed states and how the state of a system can
be determined from experiments.

Chapter V. The measuring process. Von Neumann first discusses the peculiar dual nature of the two types of quantum evolution which could not be satisfactorily explained in the previous chapters. For an undisturbed system, the state evolves causally and reversibly according to Schrödinger’s time-dependent equation, while for a system that is disturbed by a measurement, the state undergoes a non-causal and irreversible change. In the first type, pure states remain pure, while in the second type, pure states change to mixed states. Von Neumann argues that for this to be consistent, the results of measurements must be independent of the boundary line between the physical system being measured and the observer. To formulate this mathematically, he discusses the structure of composite systems in terms of their components. In modern terminology, this is given by the tensor product of the component Hilbert spaces. Applying this structure of composite systems, the book ends with a discussion of the measuring process.

This book is now in its twelfth printing. The original German text [Mathematische Grundlagen der Quantenmechanik; Springer, Berlin, 1932; JFM 58.0929.06] was published sixty-five years ago and was an elaboration of an earlier paper by von Neumann [Nachr. Ges. Wiss. Göttingen Math.-Phys. Kl. 1927, 1–57; JFM 53.0848.03]. This latter paper appeared only one year after the pioneering work of Heisenberg and Schrödinger. The present English translation was first published forty-two years ago [MR0066944]. The reviewer first read this book in 1960 and has reread parts of it at least five times. However, it has probably languished, untouched, on his bookshelf for the last twenty-five years. Rereading the text for this review has been a distinct pleasure and has brought back fond memories of excitement and discovery. How does this classic text now stand in the light of modern developments? One would expect that by now it is hopelessly out-of-date. Of course, more modern treatments have appeared. A representative sample of some of the best includes books by E. Prugovečki [Quantum mechanics in Hilbert space, Academic Press, New York, 1971; MR0495809], G. G. Emch [Algebraic methods in statistical mechanics and quantum field theory, Wiley-Intersci., New York, 1972; Zbl 235.46085], V. S. Varadarajan [Geometry of quantum theory, Second edition, Springer, New York, 1985; MR0805158] and P. Busch, M. Grabowski and P. J. Lahti [Operational quantum physics, Springer, Berlin, 1995; MR1356220]. These books are more concise, more direct, and contain simplifications in notation, methods and terminology. They each emphasize different aspects of the subject and they each carry these aspects to more distant frontiers. However, their core material and the germs of most of their ideas are contained in von Neumann’s text. Moreover, they do not contain the personality, the motivating thoughts and the seminal calculations of the master himself.  

S. Gudder