This paper is a slightly revised version of the author’s thesis from 1985. It is concerned with diffeomorphisms of the circle $T = \mathbb{R}/\mathbb{Z}$ and in particular with their centralizers and questions concerning their conjugation to rotations. We do not attempt here to sketch the history and background of these problems (starting with the work of Poincaré and Denjoy, and including important contributions by many other people, including Yoccoz’s thesis advisor Herman).

After introducing the necessary notation and recalling some basic results in Chapter I, the author studies the functional equation $\psi \circ R_\alpha - \psi = \varphi \circ R_\beta - \varphi$ in Chapter II. Here $\alpha \in T$ and $\varphi \in C^\infty(T)$ are given, $R_\alpha(x) = x + \alpha$ is the rotation, and $\beta \in T$ and $\psi \in C^\infty(T)$ are to be found. This equation is connected with the study of centralizers.

In Chapter III the conjugation to rotations is considered. Let $F_\alpha^\infty$ be the set of all $C^\infty$-diffeomorphisms of $T$ with rotation number $\alpha$ and let $O_\alpha^\infty$ be the set of all diffeomorphisms of $T$ which are $C^\infty$-conjugate to $R_\alpha$. If $\alpha$ satisfies a Diophantine condition, that is, if there exist $\gamma > 0$ and $\beta \geq 0$ such that $|\alpha - p/q| \geq \gamma q^{-2-\beta}$ for all integers $p$ and $q$, $q > 0$, then $O_\alpha^\infty = F_\alpha^\infty$. This result of the original thesis was published elsewhere [Ann. Sci. École Norm. Sup. (4) 17 (1984), no. 3, 333–359; MR0777374] and is not included here. It is shown here in Chapter III that $O_\alpha^\infty$ is dense in $F_\alpha^\infty$ with respect to the $C^\infty$-topology for each irrational $\alpha$.

In Chapters IV–VI the centralizers of diffeomorphisms of $T$ with rational rotation numbers are considered. These results are used in Chapters VII and VIII to study the centralizers of diffeomorphisms of $T$ whose rotation number is a Liouville number; that is, an irrational number that does not satisfy a Diophantine condition. One of the principal results is that there exist an irrational $\alpha$ and $f \in F_\alpha^\infty$ such that the centralizer of $f$ consists only of the iterates of $f$. For generic $f \in F_\alpha^\infty$, however, this is not the case. Moreover, for generic irrational $\alpha$ there is no $f \in F_\alpha^\infty$ with this property.

The results obtained are very important and were among the reasons why the author was awarded a Fields Medal in 1994.

\{See also MR1367353.\}

\{For the collection containing this paper see MR1367352\}