Let $T: \mathbb{Z}^d \to X$ be an action of $\mathbb{Z}^d$ by homeomorphisms of a compact metric space $X$. A continuous (real-valued) 1-cocycle for the action is a map $c: \mathbb{Z}^d \times X \to \mathbb{R}$ which is continuous in the second variable and satisfies the cocycle identity $c(n + m, x) = c(n, T_m x) + c(m, x)$. Two such cocycles $c_1$ and $c_2$ are cohomologous if there is a real-valued Borel map $b$ on $X$ for which $c_1(n, x) - c_2(n, x) = b(T_n x) - b(x)$, and are continuously cohomologous if the transfer function $b$ may be chosen continuous. A cocycle is a coboundary if it is cohomologous to zero, and is a homomorphism if $c(n, \cdot)$ is constant for each $n$.

Further regularity conditions on cocycles may be imposed in the following sense: $c$ is regular (smooth, Hölder and so on) if each map $c(n, \cdot)$ on $X$ has that property.

One of the key differences between actions of $\mathbb{Z}$ and actions of $\mathbb{Z}^d (d > 1)$ is the circle of properties loosely grouped under the heading “rigidity”. For example: an ergodic measure on the circle invariant under doubling and tripling mod 1, rendering doubling non-invertible, must be Lebesgue measure; certain natural $\mathbb{Z}^d$-actions have small measurable centralizers. The paper under review is a contribution to the cohomological aspect of this “rigidity”: roughly speaking, few non-trivial cocycles suggests a paucity of non-trivial extensions for the action.

Katok and Spatzier have shown that every real-valued Hölder 1-cocycle for an Anosov $\mathbb{Z}^d$-action on a compact manifold is Hölder cohomologous to a homomorphism. Examples of such actions are mixing expansive actions by commuting toral automorphisms. The paper under review is an extension of this result to expansive mixing $\mathbb{Z}^d$-actions by automorphisms of a compact abelian group. The requirement that the action be of this form is natural: mixing algebraic actions are faithful ones, and there is no reason to expect the result for non-expansive systems, as it seems to be intimately connected with certain specification properties.

There are several obstacles to be overcome in the extension. The groups on which $\mathbb{Z}^d$ acts expansively need not be finite-dimensional, nor locally connected. In Section 2 the authors describe natural notions of $T$-summable variation and $T$-Hölder for a continuous $\mathbb{Z}^d$-action on a compact metric space $X$. The main result (Theorem 2.1) states that if $T$ is a mixing expansive algebraic action then every 1-cocycle with $T$-summable variation is continuously cohomologous to a homomorphism, and every $T$-Hölder 1-cocycle is Hölder cohomologous to a homomorphism.

The proof involves several steps. First (Proposition 2.6) there is a technical analogue of Livshitz’ result (that a 1-cocycle for an Anosov $\mathbb{Z}$-action vanishing on periodic orbits vanishes everywhere) giving explicit obstructions to reducing a 1-cocycle to an invariant subgroup with certain specification properties.

Then the general structure of expansive mixing $\mathbb{Z}^d$-actions is invoked: each such action is determined via duality by a Noetherian module $M$ over the ring $R_d = \mathbb{Z}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$, with certain geometric constraints on the prime ideals associated with $M$ [see B. P. Kitchens and K. Schmidt, Ergodic Theory Dynam. Systems 9 (1989), no. 4, 691–735; MR1036904; K. Schmidt, Proc. London Math. Soc. (3) 61 (1990), no. 3, 480–496; MR1069512]. Decompositions from commutative algebra are then applied, allowing
an inductive reduction of a cocycle to cocycles on (finitely many) successive quotient actions under the assumption that the obstruction is checked for each quotient.

Finally (Section 4) the argument is completed by checking the vanishing of the obstruction for all the quotient actions that may arise. The Fourier series argument does not apply directly, and ingenious geometric ideas are used, along with a detailed analysis of weak specification and arithmetic hyperbolicities inside $\mathbb{Z}^d$-actions.

This is an important and satisfying result. The internal structures exhibited in expansive mixing algebraic actions will be of independent interest, so the remarks and examples throughout the course of the proof of the main result are particularly welcome.

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