This paper concerns convex inner and outer billiards in the plane. The former is a dynamical system (with discrete time) whose phase space is the set of oriented lines intersecting the billiard curve; the billiard transformation acts according to the law “angle of incidence equals that of reflection”. The latter system is a transformation of the exterior of the (outer) billiard curve $\gamma$; a point $x$ is reflected in the support point of one of the two support lines to $\gamma$ through $x$. The caustics for inner billiards studied are convex curves such that if a ray is tangent to a caustic then so is the reflected ray. The caustics for outer billiards are invariant circles of the outer billiards transformation.

The main results of the paper are quantitative versions for inner and outer billiards of the following theorem by J. Mather: The billiard transformation for a convex $C^2$ billiard table with a point of zero curvature has no invariant circles. Here is one of the main results concerning inner billiards. Theorem 1.4. Let $d$ be the diameter of a $C^2$ convex billiard table, let $k$ and $K$ be the minimal and the maximal values of its curvature, and let $L$ be the length of the billiard curve and $A$ the area of the billiard table. Assume that $\sqrt{2}d^2kK \leq 1$. Then the billiard table contains a convex region, free of caustics, whose area is at least $1 - \sqrt{2}kLd^2/A$. The outer counterpart of this theorem is the following result. Theorem 2.1. Suppose that the radius of curvature of an outer billiard curve has a zero. Then there is a Birkhoff region of instability one of whose boundaries is the outer billiard curve. (The Birkhoff region of instability for a twist map is an invariant annulus whose boundary consists of invariant circles and these are the only invariant circles in the region.) A quantitative version of this theorem, in the spirit of Theorem 1.4, is proved as well. The paper contains interesting examples of nonconvex caustics for inner billiards and of $C^1$ outer billiard curves with points of infinite curvature that have outer caustics sufficiently far away from the table.

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