Symmetric functions and Hall polynomials.
Second edition.
With contributions by A. Zelevinsky.
Oxford Mathematical Monographs.
Oxford Science Publications.

FEATURING REVIEW.
Almost immediately after appearing in 1979, the first edition of this monograph [MR0553598] became the standard reference for Schur functions (the irreducible characters of $GL_n(\mathbb{C})$), their relatives, and the rings of symmetric polynomials they generate. In 1984, A. V. Zelevinsky translated the first edition into Russian [MR0806544], and in the process, added several remarks, examples, and alternative proofs. This new edition incorporates the additions to the translated version, as well as a tremendous amount of new material. To call it a “second edition” is an understatement—it has nearly tripled in length!

Readers familiar with the first edition will find that the author’s uniquely brief but lucid style has not changed. Following the narrative portion of nearly every section is an extensive list of “Examples” (“Excursions” would be more accurate), which in fact form a catch-all category of extended remarks, exercises (sometimes together with a solution), and digressions. For example, in a tour de force display of the author’s concision, the Specht module construction of the irreducible representations of the symmetric group is derived in a single “Example” covering less than two pages (Chapter I, §7).

The first chapter concerns the fundamental bases of the ring of symmetric functions. The narrative is essentially unchanged; the main additions are an extensive collection of new “Examples”, a more complete discussion of transition matrices between various bases, and a new appendix on the irreducible characters of wreath products. The role of Schur functions as irreducible characters of $GL_n(\mathbb{C})$ is now more fully explained (including construction of the “semistandard basis” for each tensor representation), although much of it is still couched in the categorical language of “polynomial functors”.

Chapter II concerns an algebra first defined by P. Hall that encodes, via structure constants known as Hall polynomials, data regarding the lattice of subgroups of a finite abelian $p$-group (or more generally, modules over a discrete valuation ring with finite residue field). In the new edition, there is now a more explicit, but still complicated, description of the Hall polynomials, and an appendix by Zelevinsky that provides an alternative proof of some of their basic properties.

The Hall algebra of Chapter II is generically isomorphic to the ring of symmetric functions over the ground field $\mathbb{Q}(t)$, and the Hall-Littlewood functions (the subject matter of Chapter III) are the corresponding symmetric polynomials whose structure constants are the Hall polynomials. In the new version of Chapter III, there are several new “Examples” (e.g., a vertex operator construction of H-L functions), and a new section devoted to Schur’s $Q$-functions. The $Q$-functions are specializations of the H-L functions that arise naturally in several contexts, such as the projective representations of symmetric groups (mentioned only in passing), and possess a number of special
combinatorial and algebraic properties (e.g., Pfaffian identities).

Chapters IV and V, concerning the characters of GL\(_n(q)\) and the spherical functions for a Hecke algebra associated with GL\(_n\) over a local field, provide further applications of the theory of Hall-Littlewood functions. These chapters are virtually unchanged, aside from a few additional “Examples”.

Chapter VI is entirely new and contains an account of an orthogonal basis \(P_\lambda(q, t)\) of the ring of symmetric polynomials over the ground field \(Q(q, t)\) that was first presented in [I. G. Macdonald, in Actes 20e Séminaire Lotharingien, 131–171, Publ. IRMA, Strasbourg, 1988; per bibl.]. These polynomials contain as limiting or special cases the Schur functions, the Hall-Littlewood functions, and Jack’s symmetric functions, but the proofs of the main results are more difficult than the specialized counterparts. Several parts of the theory (duality, specialization, Pieri rules) are proved in the main narrative, and then again in an alternative way in the “Examples”.

Chapter VII, also new, concerns the family of symmetric functions known as “zonal polynomials”. These are the zonal spherical functions for the Gel’fand pair \((\text{GL}_n(\mathbb{R}), \text{O}_n)\), and can be obtained as a specialization of Jack’s symmetric functions. They also encode, via transition matrices, the spherical functions for the Gel’fand pair formed by the centralizer of a fixed-point-free involution in the symmetric group on \(2n\) points.

This masterful new edition will undoubtedly remain the standard reference for symmetric polynomials. It serves as proof of the meta-theorem that wherever “natural” structures indexed by partitions arise, the algebra of symmetric functions is nearby.

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