First cohomology of Anosov actions of higher rank abelian groups and applications to rigidity. (English summary)


This is the first paper in an important series of papers devoted to certain rigidity properties of Anosov actions of higher rank abelian groups. Roughly speaking, the action of an abelian group $A$ on a compact Riemannian manifold $M$ is said to be Anosov if $A$ contains an element $g \in A$ such that the tangent bundle $TM$ splits into the direct sum of stable, neutral, and unstable distributions with respect to the $g$-action, where the neutral distribution is the tangent distribution to the $A$-orbits on $M$.

Obvious Anosov actions arise as products of Anosov diffeomorphisms and flows. Here the authors study the so-called standard Anosov actions which cannot be obtained in this way. Namely, an Anosov $\mathbb{Z}^k$-action by toral automorphisms on $T^n$ is said to be standard if there is a subgroup of $\mathbb{Z}^k$ isomorphic to $\mathbb{Z}^2$ for which all nontrivial elements act ergodically on $T^n$. Clearly, this can be generalized to Anosov actions on compact nilmanifolds. Suspensions of standard Anosov $\mathbb{R}^k$-actions provide us with the first class of standard Anosov $\mathbb{R}^k$-actions. The most important example of a standard $\mathbb{R}^k$-action is the so-called Weyl chamber flow defined as follows. Let $G$ be a semisimple real Lie group with $\text{rank}_\mathbb{R} G \geq 2$, $A$ a split Cartan subgroup of $G$, $C$ the maximal compact subgroup of the centralizer $Z(A) \subset G$, and $\Gamma$ a cocompact irreducible lattice in $G$. Then the left action of $A$ on $C \setminus G / \Gamma$ is the second class of standard Anosov $\mathbb{R}^k$-actions. The third class arises if, for $G$, $A$, $C$, and $\Gamma$ as above, $G \subset \text{SL}(n, \mathbb{R})$ and $\Gamma = G \cap \text{SL}(n, \mathbb{Z})$ is an arithmetic lattice in $G$ containing an Anosov automorphism of $T^n = \mathbb{R}^n / \mathbb{Z}^n$. Then the left action of $A$ on $C \setminus G \cdot \mathbb{R}^n / \Gamma \cdot \mathbb{Z}^n$ is also Anosov (one can take here a compact nilmanifold instead of $T^n$) and is called the twisted Weyl chamber flow. Notice that all standard actions preserve volume on the corresponding compact manifold.

At the heart of rigidity properties of standard Anosov actions lies the following result. Given a standard Anosov $A$-action on $M$, where $A = \mathbb{Z}^k$ or $\mathbb{R}^k$ with $k \geq 2$, any $C^\infty$ (Hölder)-cocycle to $\mathbb{R}^l$ is $C^\infty$ (Hölder)-cohomologous to a constant cocycle. This important result is obtained via the following generalization of the well-known Livshitz theorem for Anosov flows. Let $\alpha$ be a volume-preserving Anosov $\mathbb{R}^k$-action on a compact manifold $M$ and let $\beta$ be a Hölder $\mathbb{R}^l$-cocycle over $\alpha$ such that $\beta(a, x) = 0$ for all $a \in A$ and $x \in M$ such that $ax = x$. Then $\beta$ is Hölder-cohomologous to the trivial cocycle. Another central tool for proving the main result for the (twisted) Weyl chamber flow is the exponential decay of matrix coefficients of $C^\infty$-vectors in unitary representations of semisimple Lie groups.

As applications of the main result the authors prove the following rigidity properties of standard Anosov $\mathbb{R}^k$-actions with $k \geq 2$: (1) all $C^\infty$ (Hölder)-time changes are $C^\infty$ (Hölder) conjugate to the original action up to an automorphism of $\mathbb{R}^k$; (2) any $C^1$-small Hölder perturbation is Hölder conjugate to the original action up to an automorphism of $\mathbb{R}^k$; (3) any $C^1$-small $C^\infty$-perturbation preserves volume.
It would be interesting to find all irreducible examples of rigid algebraic Anosov actions other than the standard ones given above. "Alexander Starkov"