The book presents a new approach to the problem of rational behavior. The view taken is that the problem of rational behavior cannot be regarded as a simple maximum problem, but is of a type that arises in the theory of games of strategy. The main reasons for this attitude may be briefly summarized as follows. An essential characteristic of a social-economic organization is that the participants enter into relations with each other and the result for each participant depends not only on his own actions but also on the actions of others. For example, in a social exchange economy where several persons exchange goods, the increase of utility a participant can achieve will depend on the behavior of all the other participants. In general, the situation that arises may be characterized as follows. Each participant wishes to maximize a certain function of several variables. He controls, however, only a partial set of these variables, while the values of the remaining variables depend on the actions of the other participants, who may have different or even opposite interests. The problem of a rational choice of the variables for each participant cannot be regarded as a simple maximum problem. Instead, we have a peculiar mixture of several conflicting maximum problems. Problems of the same type arise in the theory of games of strategy.

After a discussion of the nature of the problem of economic behavior, the authors turn to a development of the theory of games. The main features of the rigorous definition of the concept of a game are as follows. A game consists of a sequence of moves, $M_1, \ldots, M_\nu$. Each move $M_k$ consists of a number $\alpha_k$ of alternatives, denoted by $A_k(1), \ldots, A_k(\alpha_k)$. There are two kinds of moves, personal and chance moves. The move $M_k$ is a personal move if any of $A_k(1), \ldots, A_k(\alpha_k)$ can be freely chosen by one of the players specified by the rules of the game. The move $M_k$ is a chance move if the choice of a particular alternative is made by a chance mechanism constructed in such a way that the probability of choosing any particular alternative is equal to a value specified by the rules of the game. Denote by $\sigma_k$ the alternative chosen in the move $M_k$. The alternatives $A_k(1), \ldots, A_k(\alpha_k)$ and the number $\alpha_k$ may depend on the choices $\sigma_1, \ldots, \sigma_{k-1}$ in the preceding moves. Similarly, whether $M_k$ is a personal or a chance move, the player who makes the choice if $M_k$ is a personal move, and the probabilities assigned if $M_k$ is a chance move, may depend on $\sigma_1, \ldots, \sigma_{k-1}$. Furthermore, the total number $\nu$ of moves may itself depend on the choices $\sigma_1$, etc. The rules of the game also specify the outcome of the play (the amount each player gets when the play is completed) as a function of $\sigma_1, \ldots, \sigma_\nu$. A game is called an $n$-person game if the number of players is $n$. Denote by $F_i(\sigma_1, \ldots, \sigma_\nu)$ the outcome of the play for player $i$ ($i = 1, \ldots, n$). A game is called a zero-sum game if $\sum_{i=1}^n F_i(\sigma_1, \ldots, \sigma_\nu) = 0$ identically in $\sigma_1, \ldots, \sigma_\nu$. It is shown later that the general game can be reduced to a simple normalized form which is equivalent to the original formulation of the game. This reduction is made possible by considering the strategies of the players instead of the individual moves. By a strategy of a player is meant a complete plan which specifies what choices he should make in every possible situation. Since the number $\nu$ of moves is assumed to be bounded, the number of possible strategies for player $i$ is finite, say $\beta_i$. Then the choice of a strategy by player $i$ is characterized by the choice of a value of a variable $\tau_i$ which can take the
integral values $1, \ldots, \beta_i$. The expected value $K_i(\tau_1, \ldots, \tau_n)$ of the outcome of the game for each player can be shown to be a single-valued function of the strategies chosen by the $n$ players. The normalized form of a general $n$-person game is then as follows: $n$ functions $K_1(\tau_1, \ldots, \tau_n), \ldots, K_n(\tau_1, \ldots, \tau_n)$ and $n$ positive integers $\beta_1, \ldots, \beta_n$ are given. The variable $\tau_i$ can take the integral values $1, \ldots, \beta_i$ ($i = 1, \ldots, n$). Player $i$ chooses a particular value of $\tau_i$ and this choice is made in complete ignorance of the choices of the other $n - 1$ players. After the values $\tau_1, \ldots, \tau_n$ have been chosen, player $i$ receives the amount $K_i(\tau_1, \ldots, \tau_n)$. All further discussions are based on this normalized form of a game. The zero-sum restriction now means that $\sum_{i=1}^n K_i(\tau_1, \ldots, \tau_n) = 0$ identically in $\tau_1, \ldots, \tau_n$.

The discussion of zero-sum two-person games provides a foundation for the general theory of $n$-person games. Denote by $K(\tau_1, \tau_2)$ the outcome of the game for player 1. Then the outcome for player 2 is given by $-K(\tau_1, \tau_2)$. The fundamental problem is that of a suitable choice of the value of $\tau_1$ by player $i$ ($i = 1, 2$). Player 1 wishes to maximize $K(\tau_1, \tau_2)$ and player 2 wishes to minimize $K(\tau_1, \tau_2)$. For any particular choice of $\tau_1$, player 1 can be sure that he will get at least $\min_{\tau_2} K(\tau_1, \tau_2)$, but he cannot be sure that he will get more than this amount. Thus, if player 1 chooses a value $\tau_1'$ of $\tau_1$ for which $\min_{\tau_2} K(\tau_1, \tau_2)$ takes its maximum value, he is certain to receive an amount at least $v_1 = \max_{\tau_1} \min_{\tau_2} K(\tau_1, \tau_2)$. We shall refer to such a choice $\tau_1'$ as a minimax strategy of player 1; similarly, a choice of a value $\tau_2'$ of $\tau_2$ for which $\max_{\tau_2} K(\tau_1, \tau_2)$ takes its minimum value is a minimax strategy of player 2. If player 2 follows a minimax strategy, he can be sure to receive an amount at least $-v_2 = -\min_{\tau_1} \max_{\tau_2} K(\tau_1, \tau_2)$. If player 1 does not follow a minimax strategy, his gain may be less than $v_1$. It is shown that $v_1 \leq v_2$ always holds. A game for which $v_1 = v_2$ is called strictly determined. This case is of decisive importance, since the common value $v$ of $v_1$ and $v_2$ may be interpreted as the value of the game (for player 1) and the minimax strategies may be regarded as good strategies. If a game is strictly determined and if each of the players follows a minimax strategy, player 1 will get the amount $v$ and player 2 the amount $-v$. For strictly determined games the choice of minimax strategies by the two players creates a perfectly stable situation in the sense that even if one of the players were able to discover his opponent’s strategy, he still would have no reason to deviate from his own minimax strategy. No such stable situation exists for games which are not strictly determined.

Thus a satisfactory solution can be given only for strictly determined games; for other games there are difficulties even in defining the meaning of a good strategy. Therefore the problem is reformulated as follows. Instead of choosing a particular value of $\tau_1$, player 1 considers all possible values $1, \ldots, \beta_1$ and chooses merely the probabilities $\xi_1, \ldots, \xi_{\beta_1}$ with which he is going to use them. Similarly, player 2 chooses the probabilities $\eta_1, \ldots, \eta_{\beta_2}$ with which he is going to use the values $1, \ldots, \beta_2$ of $\tau_2$. Thus, the two choices are characterized by vectors $\xi = (\xi_1, \ldots, \xi_{\beta_1})$ and $\eta = (\eta_1, \ldots, \eta_{\beta_2})$. The expected value of the outcome $K(\tau_1, \tau_2)$ is then $K^*(\xi, \eta) = \sum_{j=1}^{\beta_2} \sum_{i=1}^{\beta_1} K(i, j) \xi_i \eta_j$. The main theorem is that the game corresponding to the outcome function $K^*(\xi, \eta)$ is always strictly determined. Player $i$ is said to have a pure strategy if he chooses a particular value of $\tau_i$, and a mixed strategy if he chooses merely the probability distribution of the possible values of $\tau_i$. Thus if both players are permitted to use mixed strategies, the game is always strictly determined. A number of special cases and examples are discussed.

The authors next turn to the discussion of zero-sum $n$-person games with $n > 2$, for which the problem has essentially new features. In fact, when $n = 2$ there is no possibility of coalitions, while when $n > 2$ some players may find it advantageous to form a coalition against the others. The possible formation of coalitions and the compensations that may be necessary between coalition partners play a decisive role. In the general theory the
formation of coalitions and the payment of compensations are governed entirely by the characteristic function of the game, defined as follows. Let \( S \) be any subset of the set \( I \) of all players, \(-S\) the complement of \( S \). Suppose that all players in \( S \) cooperate with each other, and all players in \(-S\) cooperate with each other. Then each of the groups can be regarded as one player and the game reduces to a zero-sum two-person game. Thus the game will have a definite value \( v(S) \) for \( S \). The function \( v(S) \) is called the characteristic function. A game is called inessential if for any subset \( S \) we have \( v(S) = \sum_{i \in S} v(i) \), where \( i \) denotes the subset consisting of the single player \( i \). In such a game nothing can be gained by forming coalitions and, therefore, the value of the game for player \( i \) is given by \( v(i) \). A distribution or imputation is a set of \( n \) numbers \( \alpha_1, \ldots, \alpha_n \), where \( \alpha_i \) is the amount allotted to player \( i \). For any inessential game the solution consists of the single imputation \( v(1), \ldots, v(n) \).

A game which is not inessential is called essential. For essential games coalitions and compensations play a decisive role. A solution of an essential game will be a set of imputations rather than a single imputation; furthermore, there will be, in general, several solutions. The definition of a solution is based on the notions of effective sets and domination. A subset \( S \) of the set \( I \) of all players is said to be effective for the imputation \( \alpha = (\alpha_1, \ldots, \alpha_n) \) if \( \sum_{i \in S} \alpha_i \leq v(S) \). An imputation \( \alpha \) dominates another imputation \( \beta \) if there exists a nonempty set \( S \) such that \( S \) is effective for \( \alpha \) and \( \alpha_i > \beta_i \), for all \( i \) in \( S \). A solution is defined as a set \( V \) of imputations satisfying the following two conditions. (1) No imputation in \( V \) is dominated by another imputation in \( V \). (2) Any imputation not in \( V \) is dominated by at least one imputation in \( V \). The reason that a solution \( V \) will consist of more than one imputation is that domination is not a transitive relation. A solution \( V \) may be regarded as a possible standard of rational behavior, possessing a certain inner stability which would be upset by omission of any of the two restrictions imposed on a solution. Condition (1) is needed to eliminate inner inconsistencies in \( V \). The desirability of condition (2) can be seen as follows. For any set of imputations which satisfies (1) there will be an imputation \( \beta \) outside \( V \) such that \( \beta \) dominates an imputation \( \alpha \) in \( V \). The players can hardly be expected to maintain faith in \( V \) as a standard of rational behavior unless there is another imputation \( \gamma \) in \( V \) which dominates \( \beta \). If a solution \( V \) is accepted as a standard of behavior, no imputation conforming with the accepted standard can be overruled by another. On the other hand, any imputation not conforming with the accepted standard can be overruled by an imputation conforming with this standard. There is nothing objectionable in the fact that there will usually be no unique solution. This simply means that different "established orders of society" or "accepted standards of behavior" may be built on the same physical background (the same set of rules describing a game). Each of these standards of behavior will have the characteristics of inner stability, but the different standards may well be in contradiction with each other.

The mathematical difficulties in finding solutions for games on the basis of their characteristic functions are considerable even for small \( n \). The complexity of the problem increases rapidly with increasing \( n \). Although there is hardly any doubt that every game has at least one solution, no general existence theorem has yet been established. A complete and exhaustive discussion is given for the essential 3-person game, for which all solutions are known. There is also a detailed discussion of zero-sum 4-person games. No systematic and exhaustive theory is available for \( n > 4 \). However, a number of special cases are treated, in particular, the so-called decomposable games and simple games. It is shown that all solutions of decomposable games can be determined if all solutions of the constituents are known. Furthermore, a simple game is discussed for which all solutions are determined for any value of \( n \).

In dealing with problems of behavior in a social economy, the zero-sum restriction is
not very realistic. An extension of the theory to general non-zero-sum games is achieved by re-interpreting an \( n \)-person general game \( \Gamma \) as an \((n+1)\)-person zero-sum game in the following way. Let \( K_i(\tau_1, \cdots, \tau_n) \) \((i = 1, \cdots, n)\) be the outcome functions of the \( n \)-person general game \( \Gamma \). A fictitious player \((n+1)\) is introduced by defining \( K_{n+1}(\tau_1, \cdots, \tau_n) = -\sum_{i=1}^{n} K_i(\tau_1, \cdots, \tau_n) \). In this way we obtain an \((n+1)\)-person zero-sum game \( \Gamma' \), which is called the zero-sum extension of \( \Gamma \). It is shown that the only solutions of \( \Gamma' \) which can be interpreted as solutions of \( \Gamma \) are those which are in accordance with the fact that the fictitious player cannot have even an indirect influence on the course of the game. It turns out that a solution \( V \) of \( \Gamma' \) can be regarded as a solution of \( \Gamma \) if for any imputation \( \alpha = (\alpha_1, \cdots, \alpha_{n+1}) \) in \( V \) we have \( \alpha_{n+1} = v(n+1) \). The solutions of all general games with \( n \leq 3 \) are derived and the results are applied to a social exchange economy with 2 and 3 participants. These economic applications deserve special attention, since even in these simple cases the new approach deviates from the ordinary viewpoint. Consider first the case of two participants: 1, the seller; 2, the buyer. Suppose that the transaction is the sale of one unit \( A \). Let \( u \) denote the value of \( A \) to 1 and \( v(v > u) \) to 2. Then the commonsense result is that the price \( p \) may be anywhere between the two limits \( u \) and \( v \). The theory of games leads to exactly the same result. However, if the conditions of the problem are changed by allowing transactions involving any or all of \( s \) units of a commodity \( (s > 1) \), the result implied by the theory of games will differ from the commonly accepted viewpoint as expressed, for example, in the Boehm-Bawerk theory. While the number of units transferred according to the theory of games will be the same in both theories, the Boehm-Bawerk theory restricts the price to a narrower interval. The reason is that the ordinary viewpoint assumes the existence of a unique price valid for all transfers, while the theory of games allows for premiums and rebates (compensations) which obliterate the unique price. Another qualitative discrepancy occurs in the case where one seller, 1, sells a unit \( A \) to either buyer 2 or buyer 3. Let \( u \) be the value of \( A \) for 1, \( v \) for 2 and \( w \) for 3, where \( u < v \leq w \). The solution \( V \) given by the theory of games is the sum of the following two sets of imputations: the set of all imputations \((\alpha_1, \alpha_2, \alpha_3)\) which satisfy the condition \( (*) \ v \leq \alpha_1 \leq w; \ \alpha_2 = 0; \ \alpha_3 = w - \alpha_1 \) and the set of imputations for which \( (**) \ u \leq \alpha_1 \leq v; \ \alpha_2, \ \alpha_3 \) are certain monotonic decreasing functions of \( \alpha_1 \). In the ordinary treatment only imputations satisfying \( (*) \) are considered. The imputations satisfying \( (**) \) can be interpreted as arising from a coalition of the two buyers against the seller. Also, the case of divisible goods is discussed for the 3-person market. Some remarks are made about the general \( n \)-person market.

The book will be of interest to sociologists and economists who are interested in a rigorous foundation of the theory of rational behavior, in particular, the theory of exchange in a general market. [It may be of interest to mention that the theory of games has applications to statistics as well, since the general problem of statistical inference may be treated as a zero-sum two-person game.] The authors make an endeavor to present the material in a form which will make it understandable to readers without a special knowledge of any part of advanced mathematics. In spite of this, the book is not really elementary, because of the intricate nature of many of the mathematical deductions. Nonmathematical readers will find the excellent verbal expositions given parallel with every major mathematical deduction very helpful.

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