Local rigidity for certain groups of toral automorphisms.


Structural stability for a dynamical system, given by a differentiable map $F: X \rightarrow X$, states that a nearby mapping $G$ in the $C^1$-topology on maps is equivalent to $F$: that is, there is a topological conjugacy $H: X \rightarrow X$ conjugating $F$ to $G$. S. Smale's fundamental paper [Bull. Amer. Math. Soc. 73 (1967), 747–817; MR0228014; MR errata, EA 39] addressed various aspects of the problem of showing that a map is structurally stable, but also introduced the same question for group actions. Almost 20 years later, R. J. Zimmer [in *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)*, 1247–1258, Amer. Math. Soc., Providence, RI, 1987; MR0934329] posed the related problem of showing that ergodic actions of higher lattices on compact manifolds all arise from algebraic constructions, which implies a much stronger form of stability for these actions. As a special case, it was asked whether the standard action of the group of determinant one matrices $SL(n, \mathbb{Z})$ on the $n$-torus $\mathbb{T}^n$ is structurally stable. In the paper under review, the authors prove that for a subgroup $\Gamma \subset SL(n, \mathbb{Z})$ of finite index for $n \geq 4$, its standard linear action on the $n$-torus $\mathbb{T}^n$ is structurally stable. Their methods also apply for a subgroup of finite index of the symplectic matrices $Sp(n, \mathbb{Z}) \subset SL(2n, \mathbb{Z})$ for $n \geq 3$, and for products of such groups. The proof of this theorem combines the algebraic structure of $\Gamma$ with the dynamics of the action. The authors' key observation is that $\Gamma$ as above admits a sufficiently rich collection of subgroups whose actions are normally hyperbolic to linear foliations on $\mathbb{T}^n$ with all leaves compact. These subactions satisfy structural stability along the leaves of the foliation, from which the authors derive the uniform structural stability for the periodic points of the full action. This suffices to prove the claim of the theorem, as shown by the reviewer [Ann. of Math. (2) 135 (1992), no. 2, 361–410; MR1154597]. The authors’ approach to stability can also be extended to obtain global rigidity results that a given Anosov action with appropriate cohomological hypotheses must be conjugate to an affine action [A. Katok and J. Lewis, “Global rigidity results for lattice actions on tori and new examples of volume-preserving actions”, Preprint; per bibl.]. Note that global stability does not hold, as there are perturbations of linear actions which have no fixed points [S. Hurder, “Affine Anosov actions”, Michigan Math. J., to appear].

Anosov actions of sufficiently large lattice actions often enjoy an additional rigidity property that a topological conjugacy must be as smooth as the actions [Hurder, op. cit., 1992]. For the case of $\Gamma$ discussed in the paper under review, if the perturbed action is either smooth or analytic, then it is conjugate to the linear action by a smooth or analytic diffeomorphism. The authors give a new proof of this property in the smooth case, replacing the reviewer’s use of lattice rigidity with intrinsic rigidity for the Gibbs measures along the leaves of the invariant foliations for the action. The last part of the paper uses this idea to prove the following remarkable result: A free abelian Anosov smooth action of “maximal rank” $n - 1$ on $\mathbb{T}^n$ with a common fixed-point is smoothly conjugate to a linear action. This smoothness principle admits many other applications (cf. related papers by E. E. Cawley [Internat. Math. Res. Notices 1992, no. 7, 135–141;...
MR1174618] and A. Katok and R. Spatzier [“Differential rigidity of hyperbolic abelian actions”, Preprint, 1992; per revr.].

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