Weighted shift operators and linear extensions of dynamical systems. (Russian)


The paper is devoted to the class of so-called weighted composition operators. It consists of an introduction, four sections, and a comprehensive bibliography gathered at the end of the paper. Our review consists of straight translations of suitable parts of the paper. We always indicate the section in which quoted results belong. We stress that weighted composition operators often are called weighted translation (shift) operators in the literature.

Let $X$ be an abstract set. A weighted composition operator (WCO) $Q$ is an operator acting in the space of vector-valued functions $f$ (defined on $X$) as: $Qf(x) = q(x)f(\alpha(x))$ where $x \in X$, $q$ is a fixed operator-valued function and $\alpha$ is a mapping of $X$ into itself. This kind of operator may be found in several branches of modern analysis, e.g., in the theory of dynamical systems, theory of operator ($C^*$) algebras, theory of differential equations, etc. The main aim of the paper is to study relations between the spectral theory of WCOs, continuous semigroups of such operators and the theory of infinite-dimensional linear extensions of dynamical systems. $C^*$ algebra methods are efficiently used to study linear extensions. And vice versa, the dynamical systems approach to WCOs gives valuable results. The multiplicative ergodic theorem is exploited to describe the spectra of WCOs as well as their extensions. After this brief summary we present a few representative results of the paper. We start with introductory notations and definitions.

By $\alpha = \{\alpha^t\}$ is denoted a flow ($t \in \mathbb{R}$) or a cascade ($t \in \mathbb{Z}$) on a compact metric space $X$ equipped with a quasi-invariant Borel measure $\mu$ (i.e. $\mu(E) = 0$ if and only if $\mu(\alpha^t(E)) = 0$ and $\mu(U) > 0$ for any open set $U \subseteq X$). $A: X \times \mathbb{R}_+ \to \mathcal{L}(H)$ or $A: X \times \mathbb{Z}_+ \to \mathcal{L}(H)$ denotes an $\mathcal{L}(H)$-valued $\alpha$-cocycle, where $\mathcal{L}(H)$ is the algebra of bounded operators on a Hilbert vector space $H$. The authors introduce a semigroup $\{T_t\}$ of WCOs acting in $L_2 = L_2(X, \mu, H)$ by the formula

\[ (*) \quad T_{\alpha^t}f(x) = \left( \frac{dm \cdot \alpha^{-1}t}{dm} \right)^{1/2} A(\alpha^{-t}(x), t)f(\alpha^{-t}(x)) \]

where $f \in L_2$, $x \in X$. They also consider a linear extension $\hat{\alpha} = \{\hat{\alpha}^t\}$ which is a semigroup of mappings of $X \times H$ into itself defined as $\hat{\alpha}^t(x, v) = (\alpha^t(x), A(x,t)v)$ where $t \in \mathbb{R}_+$ or $\mathbb{Z}_+$. For any operator-valued functions $a: X \to \mathcal{L}(H)$, a WCO $T_a$ is defined as $T_a f(x) = \left[ dm \cdot \alpha^{-1}/dm \right]^{1/2} a(\alpha^{-1}(x)) f(\alpha^{-1}(x))$ where $x \in X$ and $f \in L_2(X, m, H)$. Clearly the formula $A(x, 0) = \mathbb{I}$, \cdots $A(x, n) = a(\alpha^{-n}) \cdots a(x)$, where $n \in \mathbb{Z}_+$, $x \in X$, defines a cocycle. The general exponent $\kappa_\gamma$ of a cocycle $A$ is defined as the infimum of all $\gamma \in \mathbb{R}$ such that $\|A(x, n)v\|_H \leq C_\gamma e^{\gamma(n-k)} \|A(x, k)v\|_H$ holds for all $v \in H$, $x \in X$, $k, n \in \mathbb{Z}_+$ and $0 \leq k \leq n$.

(1) In the first section the authors present (Proposition 1.1) the following formula:

$\kappa_\gamma = \ln(\lim_{n \to \infty} \max_{x \in X} \|A(x, n)\|^{1/n})$. Next are results about spectral properties of operators of the form $T_a$. It is proved that if $\mu(\text{Per } \alpha) = 0$ (where $\text{Per } \alpha$ denotes the set of all periodic points of $\alpha$) then $\sigma(T_a)$ is invariant under rotations (Proposition 1.2).
the other hand, if $\alpha$ is a periodic mapping with period $n$ (i.e. $\alpha^n(x) = x$ and $\alpha^j(x) \neq x$, $j = 1, \ldots, n - 1$, where $x \in X$), then $\sigma(T) = \{\lambda: \lambda^n \in \bigcup_{x \in X} \sigma(A(x, n); H)\}$.

(2) Next continuous semigroups are considered. Let $\{\alpha_t\}_{t \in \mathbb{R}}$ be a flow of homeomorphisms of $X$. Consider a semigroup defined in $(\ast)$. It is proved (Proposition 1.5) that if $\{T^t_A\}$ is a strongly continuous semigroup and for any $v \in H$ the convergence $\lim_{t \to 0} \sup_{x \in X} \|A(x, t)v - v\|_H = 0$ holds then the semigroup $\{T^t_A\}$ is also strongly continuous. The crucial role in the proofs of the main results of this item is played by the following version of Gearhart’s theorem which is formulated as Lemma 1.8: Let $\{e^{tD}\}_{t \geq 0}$ be a strongly continuous semigroup of operators acting on a Hilbert space. Then $\sigma(e^{tD}) \setminus \{0\} = \{e^{\lambda t}: \lambda = \lambda + 2\pi i k/t \in \sigma(D)\}$ for some $k \in \mathbb{Z}$ or the sequence $\|R(\omega_k, D)\|_{k \in \mathbb{Z}}$ is unbounded where $t > 0$ and $R(z, D)$ is the resolvent of an operator $D$. Subsequently the authors define the dynamical spectrum $\sigma$ and the spectrum $\sigma(D)$ that if for any $\omega$ for which all extensions $\hat{\alpha}$ satisfies $\text{ess inf} \{\lambda \in \mathbb{R} \mid \sigma(T^\lambda_A) \neq 0\}$ implies that the spectrum of $D$ is translation invariant “along the imaginary axis” and $\sigma(T^\lambda_A) \setminus \{0\} = \mathcal{F}(e^{i\lambda \sigma(D)}), t \geq 0$. On the other hand (Theorem 1.11), if the function $p$ of periods satisfies $\text{ess inf} p(x) > 0$ then $\sigma(T^\lambda_A) \setminus \{0\} = \mathcal{F}(e^{i\lambda \sigma(D)}), t \geq 0$ where $\mathcal{F}(\cdot)$ denotes the union of all concentrated circles which meet a set $(\cdot)$.

Subsection 1.3 consists of valuable remarks and supplements related to the problems presented above.

In Section 2 preliminary facts about noncommutative dynamical systems are collected.

In the subsequent Section 3 the authors study relations between WCOs and spectral theory of linear extensions.

(3) Let $\theta_x$ denote the class of “elementary” weighted shift operators defined on $l_2 = l_2(\mathbb{Z}, H)$ as $\theta_x: (v_n)_{n \in \mathbb{Z}} \rightarrow (\alpha^n(x)v_n)_{n \in \mathbb{Z}}, x \in X$. It is proved (Theorem 3.2) that an operator $T_\alpha$ is hyperbolic in $L_2$ if and only if the extension $\hat{\alpha}$ is hyperbolic and if and only if the spectra of discrete operators $\theta_x$ in $l_2$ are separated from the unit circle uniformly on $X$. We recall that a linear extension of $\hat{\alpha}$ is called hyperbolic on $X$ (Definition 3.1) if there exists a continuous projector-valued function $P: X \rightarrow \mathcal{L}(H)$ such that $A(x, n)P(x) = P(\alpha^n(x))A(x, n)$ for all $x \in X$, $n \in \mathbb{Z}_+$ and moreover there are constants $C > 0$, $\beta > 0$ such that for all $n \in \mathbb{Z}_+$ and $x \in X$ we have: $\|A(x, n)v\| \leq Ce^{-\beta n}\|v\|$ if $v \in \overline{\text{Im} P(x)}$ and $\|A(x, n)v\| \geq Ce^{\beta n}\|v\|$ if $v \in \overline{\text{Im}(I - P(x))}$ (here $A(x, n)$ denotes a cocycle defined by $a$ and $\alpha$). The above results are applied to some class of discrete Schrödinger operators $L_\omega = V + V^{-1} + \text{diag}\{q(\alpha^n(x))\}_{n \in \mathbb{Z}}$ Id acting on $l_2(\mathbb{Z})$. Subsequently the authors define the dynamical spectrum of an extension $\hat{\alpha}$ as the set $\Sigma = \Sigma(\hat{\alpha})$ of all $\omega \in \mathbb{R}$ for which all extensions $\hat{\alpha}_\omega(x, v) = (ax, e^{-\omega}a(x)v)$ are not hyperbolic on $X$. It is proved (Proposition 3.6) that $\Sigma = \ln \{\sigma(T_\alpha)\}$.

3.2 Similar to the discrete case, the spectral theory of WCO semigroups with real time parameter is developed. First the concept of hyperbolicity of extensions is introduced (quite analogously as in the discrete case). Next it is proved (Proposition 3.10) that the hyperbolicity of $\{\hat{\alpha}_t\}_{t \in \mathbb{R}}$ is equivalent to the hyperbolicity of $\{\alpha^n\}_{n \in \mathbb{Z}}$. In Corollary 3.11 it is noticed that the hyperbolicity of $\{\hat{\alpha}_t\}_{t \in \mathbb{R}^+}$ is equivalent to the hyperbolicity of the generator $\mathcal{D}_A$ of the semigroup $\{T^t_A\}_{t \in \mathbb{R}^+}$. Finally (Proposition 3.12) it is shown that the dynamical spectrum $\Sigma$ and the spectrum $\sigma(\mathcal{D}_A)$ of the generator of $\{T^t_A\}_{t \in \mathbb{R}^+}$ satisfy $\Sigma = \sigma(\mathcal{D}_A) \cap \mathbb{R}$.

3.3 In this subsection so-called continual operators are considered. According to the notation, continual operators are WCO’s $\Theta_x^t$ acting on $L_2(\mathbb{R}, H)$ which act as $\Theta_x^t f(s) = A(\alpha^{s(t)}(x), t)f(s - t)$ where $s \in \mathbb{R}$ and $f \in L_2(\mathbb{R}, H)$. It is proved (Proposition 3.13) that if for any $v \in H$ we have $\lim_{t \to 0} A(x, t)v = v$ in the $H$-norm and uniformly on $X$ then for all $x \in X$ the semigroup $\Theta_x^t$ is strongly continuous. The subsequent result (Theorem 3.16) establishes relations between spectra. It is proved that under the condition that for any $x \in X$ the semigroup $\{\Theta_x^t\}_{t \in \mathbb{R}^+}$ is strongly continuous on $L_2(\mathbb{R}, H)$, the spectrum
σ(dx) of its generator dx is invariant along the imaginary axis and \( \sigma(\Theta^t_x) \setminus \{0\} = e^{t\sigma(dx)} \).

The item ends with the following (Theorem 3.17): The extension \( \{\hat{\alpha}^t\}_{t \in \mathbb{R}_+} \) is hyperbolic if and only if the spectrum of the operator \( \Theta^t_x \) is separated from the unit circle uniformly on \( X \).

3.4 Now let us recall so-called exponential dichotomy (Definition 3.20). A linear extension is said to be exponentially dichotomic at the point \( x \in \mathbb{X} \) if there exist a projection \( P(x) \in \mathcal{L}(H) \) and constants \( C, \beta > 0 \) such that
\[
\|A(x, t)P(x)A^{-1}(x, s)\| \leq Ce^{-\beta(t-s)} \quad \text{for} \quad -\infty < s \leq t < \infty \quad \text{and} \quad \|A(x, t)(I - P(x))A^{-1}(x, s)\| \leq Ce^{-\beta(s-t)} \quad \text{for} \quad -\infty < t < s \leq \infty
\]
(here \( s, t \in \mathbb{Z} \) or \( s, t \in \mathbb{R} \)). It is proved that for exponential dichotomy of an extension \( \{\hat{\alpha}^t\} \) at the point \( x_0 \in \mathbb{X} \) it is sufficient that the operator \( dx_0 \) be invertible and for all \( x \in \{\alpha^t(x_0) \mid t \in \mathbb{R}\} \) the operators \( dx \) be hyperbolic.

3.5 As usual, the section ends with a discussion of known results related to the included ones.

In the final Section 4, properties of the dynamical spectrum \( \Sigma \) are studied.

4.1 Using the multiplicative ergodic theorem the geometric structure of \( \Sigma \) is described (Theorems 4.5 and 4.6).

4.2 In this subsection a formula to determine the spectral radius \( R(T_a) \) through Lyapunov exponents is established (Theorem 4.7).

4.3 A few methods for computing or estimating the spectral radius of a WCO generated by an endomorphism are presented.

Finally the role of the multiplicative ergodic theorem in this analysis is discussed.

Wojciech Bartoszek

© Copyright American Mathematical Society 2018