One of the most widely studied classes of dynamical systems is that of diffeomorphisms of the circle. The behaviour of such a map \( f \) is not on the whole well understood. It may be classified in terms of the rotation number \( \rho(f) \) which, roughly speaking, measures the average rate of rotation of orbits around the circle. When \( \rho(f) = \frac{p}{q} \) is rational, \( f \) will have at least one periodic orbit of period \( q \) (and generically at least two) and all other orbits will converge to some \( q \)-periodic orbit both forward and backward in time. On the other hand, when \( \rho(f) = \alpha \) is irrational, \( f \) has no periodic orbits and its dynamics depends on the smoothness of \( f \). Thus, A. Denjoy [J. Math. Pures Appl. (9) 11 (1932), 333–375; Zbl 6, 305] proved that if \( f \) is \( C^2 \) then it is conjugate to the rotation \( R_{\alpha} (\theta) = \theta + \alpha \mod 1 \), i.e. there is a homeomorphism \( h \) of the circle such that \( h \circ f = R_{\alpha} \circ h \). Since then a succession of authors have shown that if \( f \) is smoother than \( C^2 \) and \( \alpha \) satisfies appropriate Diophantine estimates then \( h \) will have some degree of smoothness. The strongest results can be found in other papers [Y. Katznelson and D. Ornstein, Ergodic Theory Dynamical Systems 9 (1989), no. 4, 643–680; MR1036902; Ya G. Sina˘ı and Khanin, Uspekhi Mat. Nauk 44 (1989), no. 1(265), 57–82; MR0997684; the reviewer, Nonlinearity 1 (1988), no. 4, 541–575; MR0967471] which also contain references to previous work. The last two of these references depart from previous work in that they make use of renormalization techniques. These are rapidly coming to be seen as powerful tools for the study of a wide variety of phenomena in dynamical systems.

The present paper is an extension of the work carried out by Sinai and Khanin [op. cit.]. The authors consider a generalization of a diffeomorphism of the circle in which the map has a single discontinuity in its first derivative. Such a generalization arises quite naturally within the renormalization framework, partly because the ratio of the two slopes at the discontinuity is essentially invariant under renormalization. Such circle maps have also arisen in the study of implicit complex maps of the plane [B. D. Mestel and A. H. Osbaldestin, Phys. D 39 (1989), no. 2-3, 149–162; MR1028712] where they are called almost \( C^1 \) maps of the circle.

The main result of the present paper is that under renormalization any such map converges to a two-dimensional space of fractional linear transformations. As far as the reviewer is aware, this is the first rigorous proof of the existence of a nontrivial attractor for a renormalization group, though there has been numerical evidence for some time now for the existence of such attractors for a variety of renormalization operators [see, e.g., D. Rand, in New directions in dynamical systems, 1–56, Cambridge Univ. Press, Cambridge, 1988; MR0953969]. The authors then use this reduction to a finite-dimensional subspace to prove two important results concerning such maps with rational rotation number. First, they show that such a map has at most two periodic orbits, and second, that the set of parameter values \( \omega \) for which the one-parameter family of maps \( f_\omega = f + \omega \) has rational rotation number has full measure. This last result is proved in a similar fashion to the analogous statement for critical circle maps (that is, maps which are diffeomorphisms except at one point where the derivative is...
1, 109–128; MR0968483].

{For the collection containing this paper see MR1118155}

Jaroslav Stark

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