A study of a numerical solution to a two-dimensional hydrodynamical problem.


The authors consider non-steady motion under gravity of two incompressible fluids in a rectangular cell. Initially the fluids are at rest, with fluid of density $\rho = 1$ in a trapezoid that forms the upper portion of the cell, and fluid of density $\rho = 1/8$ in the complementary lower trapezoid. The equations of motion and continuity are written in Eulerian form and are supplemented by (i) $\rho_t + u\rho_x + v\rho_y = 0$. The stream function $\psi$ is introduced and pressure is eliminated to produce a system of two partial differential equations for $\psi$ and $\rho$. These are solved numerically by converting them to difference equations. An essential feature of the motion is the density discontinuity at the interface between the two fluids, but this was not explicitly taken into account in the choice of difference equations. Accordingly, in the course of calculation the discontinuity gradually becomes spread out, a phenomenon described as “pseudo-diffusion” by von Neumann.

One purpose of the work was to see if the motion of the interface could be followed in time from a plot of density contours.

Flow charts and detailed statements of difference equations are given. The principal novelty is the treatment of the density equation (i). In the body of the fluid several alternative forms were used, the choice depending on the signs of $u$ and $v$. At the boundaries of the cell similar difference versions of $\rho_t + (\rho u)_x + (\rho v)_y = 0$ were used, since the difference forms of (i) do not allow the discontinuity to reach one boundary of the cell. A study of the equation $\rho_t + u\rho_x = 0$ of one-dimensional flow suggests the stability condition $\Delta t \leq \min (\Delta x/|u|, \Delta y/|v|)$. Computations were carried out on a $15 \times 38$ lattice. In graphs of computed results the interface was taken to be the curve of mean $\rho = 0.5625$, and the boundaries of the “pseudo-diffusion” band were taken as $\rho = 0.5625 + 0.1750$ (20% original density difference). At the 60th time step (0.339 sec) the band was generally less than two space meshes ($\Delta x = \Delta y = 1$ cm) wide.

The essence of the numerical work is described in von Neumann’s notes on a scheme for accelerating iterative solutions of systems of linear equations (ii) $A\xi = \alpha$ that arise from partial differential equations of elliptic type. A typical iterative method could be taken in the form

$$(\xi_{k+1}^\alpha) = (G \ H \ O \ I) \ (\xi_k^\alpha) \equiv E \ (\xi_k^\alpha),$$

where $G$, $H$, $O$, and $I$ (unit) are square matrices of the same order as $A$, and $\xi_k^\alpha$ is a $k$th approximation to $\xi$. (iii) conserves the solution of (ii) if and only if $G = I - HA$, where $H$ is non-singular. To accelerate convergence von Neumann proposed to use as $k$th approximation a mean $n_k^\alpha = \sum_0^k \alpha_k \xi^l$, or

$$(n_k^\alpha) = P_k(E) \ (\xi_0^\alpha), \quad P_k(E) = \sum_0^k \alpha_k E^l.$$

Plausible arguments are given to suggest that most rapid convergence will be achieved if $G$ is taken to be a Hermitian matrix with characteristic roots between $-1 + \varepsilon$ (min)
and $1 - \varepsilon$ (max), and if $P_k(E)$ (aside from elementary transformations) are taken to be Chebyshev polynomials. It is shown that (iv) converges $(2/\varepsilon)^{0.5}$ times as fast as ordinary iteration (iii). In general von Neumann suggests taking $H = \alpha A^*$, where $*$ denotes conjugate transpose. However, for an appropriate difference scheme for an elliptic partial differential equation for which $A$ is hermitian, take $H = \alpha I$. In either case, the scalar $\alpha$ is used to shift the characteristic roots of $G$ within the desired range.

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References


Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

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