In 1846 C. G. J. Jacobi invented a direct and constructive method for proving that a symmetric matrix can be diagonalized by orthogonal transformations. He uses as elementary transformations 2-dimensional rotations in planes spanned by 2 coordinate axes. Or—geometrically speaking—the ellipsoid represented by the given matrix is introduced and its plane section by such a coordinate plane is examined. Then the 2 corresponding coordinate axes are rotated into the principal axis of this plane section. Observing that the sum of the squared elements of the matrix is invariant under orthogonal transformations he shows that the sum $\tau$ of the squared off-diagonal elements is diminished by each rotation and vanishes in the limit. Later on, Runge recommended this method as a numerical procedure for computing eigenvalues of symmetric matrices. The paper under consideration is an extensive study of such a procedure taking into account convergence properties and rounding-off behaviour as well. In the introduction the authors discuss the numerical difficulties inherent in all methods for computing the characteristic polynomial of a given matrix and of finding the eigenvalues by computing the roots of this polynomial. Therefore direct methods—for instance Jacobi’s method—should be preferred. They state the problem, introduce a sequence $B^{(i)}$ of matrices obtained by groups of Jacobi rotations and give an estimate of the deviations of the diagonal elements of $B^{(i)}$ from the wanted eigenvalues based upon an estimate of the sum $\tau_i$ corresponding to $B^{(i)}$. They show that the Jacobi algorithm can be established in such a way that $\tau_i \leq \exp(-i)$ and that the diagonal elements of $B^{(i)}$ are within $\exp(-i/2)$ of the true eigenvalues.

But the outstanding feature of the paper is a thorough discussion of the rounding-off properties of the algorithm. The authors use the same tools as in the famous paper on inverting of matrices of high order [Bull. Amer. Math. Soc. 53 (1947), 1021–1099; MR0024235] by J. von Neumann and H. H. Goldstine. They deal with pseudo-operations of a computing instrument and discuss in particular the pseudo-square root and pseudo-trigonometric functions as components of the Jacobi algorithm. This enables them to give at the end of their paper explicit estimates of the rounding-off errors to be expected by computing eigenvalues using Jacobi’s method. Very few algorithms in the field of numerical analysis have been studied so thoroughly as is Jacobi’s method in this paper.

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