Differentiability of entropy for Anosov and geodesic flows.


The paper contains theorems due to various combinations of the authors. The topological entropy of a flow \( \{ \varphi^t \} \) is defined to be the topological entropy, \( h_{\text{top}}(\varphi^1) \), of the time-one map. Consider an Anosov flow \( \{ \varphi^t \} \) of a compact manifold \( M \) and consider a family \( \{ \varphi^t_\lambda \} \) of perturbations of it \((\varphi^t_0 = \varphi^t)\). If all the flows are \( C^\omega \) (real analytic) and \( \lambda \to \{ \varphi^t_\lambda \} \) is \( C^\omega \), then \( \lambda \to h_{\text{top}}(\varphi^1_\lambda) \) is \( C^\omega \). If all the flows are \( C^{k+1} \) and \( \lambda \to \{ \varphi^t_\lambda \} \) is \( C^{k+1} \) and \( 1 \leq k \leq \infty \), then \( \lambda \to h_{\text{top}}(\varphi^1_\lambda) \) is \( C^{k} \). However, when the flows are \( C^1 \) and \( \lambda \to \{ \varphi^t_\lambda \} \) is \( C^1 \), then \( \lambda \to h_{\text{top}}(\varphi^1_\lambda) \) is \( C^{1} \). In this last case

\[
\left. \frac{\partial h_{\text{top}}(\varphi^1_\lambda)}{\partial \lambda} \right|_{\lambda=0} = h_{\text{top}}(\varphi^1) \left[ \int_M \frac{\partial \alpha_\lambda(p)}{\partial \lambda} \bigg|_{\lambda=0} d\mu_0(p) \right],
\]

where \( \mu_0 \) is the unique measure of maximal entropy for \( \{ \varphi^t \} \) and \( \alpha_\lambda(p) \) denotes a function which gives the time reparametrization in structural stability when integrated along orbits of \( \{ \varphi^t \} \).

In the case of geodesic flows one can obtain another formula. If \( g \) is a Riemannian metric of negative sectional curvature on \( M \) and if \( \lambda \to g_\lambda \) is a \( C^2 \) map from \((-\epsilon, \epsilon)\) into the space of \( C^2 \) metrics on \( M \) with \( g_0 = g \), then if \( \{ \varphi^t_\lambda \} \) denotes the geodesic flow corresponding to \( g_\lambda \) we get that \( \lambda \to h_{\text{top}}(\varphi^1_\lambda) \) is \( C^{1} \) and

\[
\left. \frac{\partial h_{\text{top}}(\varphi^1_\lambda)}{\partial \lambda} \right|_{\lambda=0} = -\frac{h_{\text{top}}(\varphi^1)}{2} \int_{SM_0} \frac{\partial g_\lambda(v,v)}{\partial \lambda} \bigg|_{\lambda=0} d\mu_0(v),
\]

where \( SM_0 \) is the unit tangent bundle for \( g \) and \( \mu_0 \) is the unique measure of maximal entropy for \( \{ \varphi^t_0 \} \). If \( R_1(M^2) \) is the submanifold of negatively curved metrics on a compact surface \( M^2 \) with area equal to 1, then \( h_{\text{top}}: R_1(M^2) \to \mathbb{R} \) has a critical point at \( g \) if and only if \( g \) is a metric of constant negative curvature.

Some results are also given for metric entropy. All proofs are given elsewhere.

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